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Sums of Sums of Power Series

by Jim Milner

Prelude

The starting point for this work was collaborative work which I did with Professor Jan van Maanen of Utrecht University during February and March 2019. We were looking at ways that Archimedes could have discovered his formula for the sum of the squares and, also ways that the Babylonians could have independently discovered their very different formula for the same sum, which appears on the Babylonian clay tablet AO 6484. Archimedes and Anu-aba-uter, the scribe who inscribed AO 6484, were roughly contemporaneous. We plan a joint paper about this work, but it has not been published yet. The diagrams of the sum of the squares and the sum of the sum of the integers on pages 2 and 4 are essentially Jan's work not mine (See *diagram 2* on page 2 and *diagram 6* on page 4).

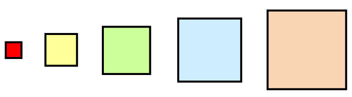
In the following pages, I introduce this notation:

$$\sum_{i=1}^{n} i^p = \left({}^{S=1}\sum_{i=1}^{n} i^p \right) \qquad \sum_{j=1}^{n}\sum_{i=1}^{j} i^p = \left({}^{S=2}\sum_{i=1}^{n} i^p \right) \qquad \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} i^p = \left({}^{S=3}\sum_{i=1}^{n} i^p \right) \text{... etc...}$$

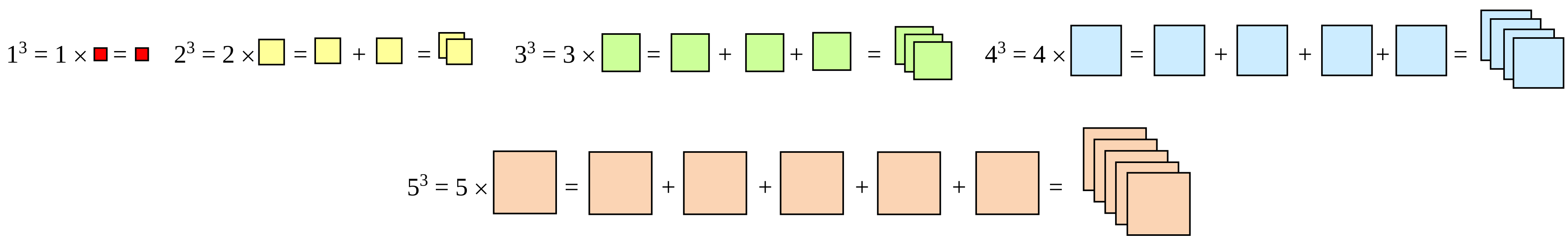
I hope the reader will soon become accustomed to it.

Key to the diagrams

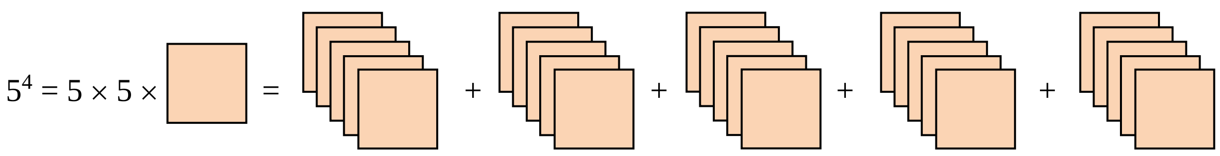
All the powers in this work are represented as occupying two dimensional spaces. For example, the integers, which are one dimensional, have been promoted to being two dimensional by representing 1 as the square 1 × 1, 2 as the rectangle 1 × 2, and so on:

The squares are two dimensional. Here are the squares up to 5^2: 

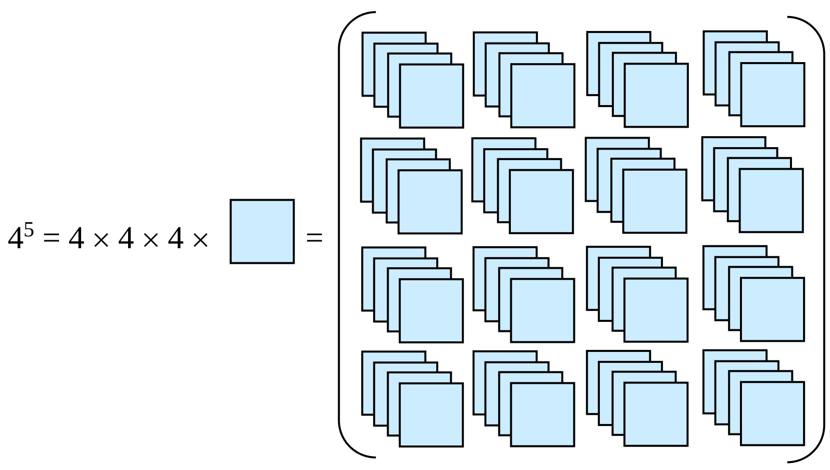
The cubes are represented as being two dimensional by the expedient of regarding n^3 as being equal to $n \times n^2$. Here are the first five cubes:



n^4 is equal to $n \times n \times n^2$:

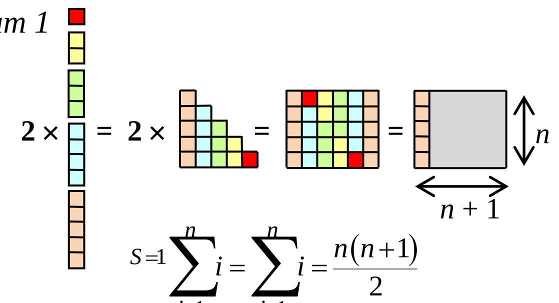


n^5 is equal to $n \times n \times n \times n^2$:



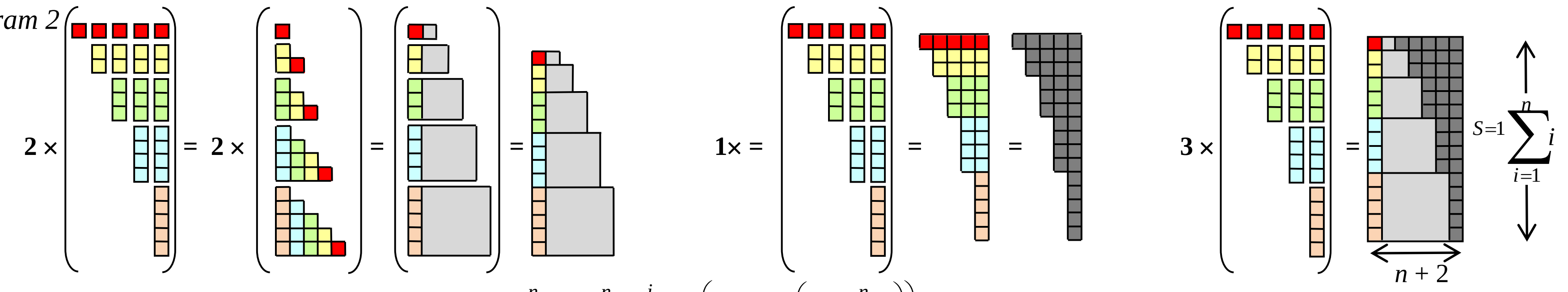
Sums of integers

Diagram 1



$$^{S=1}\sum_{i=1}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

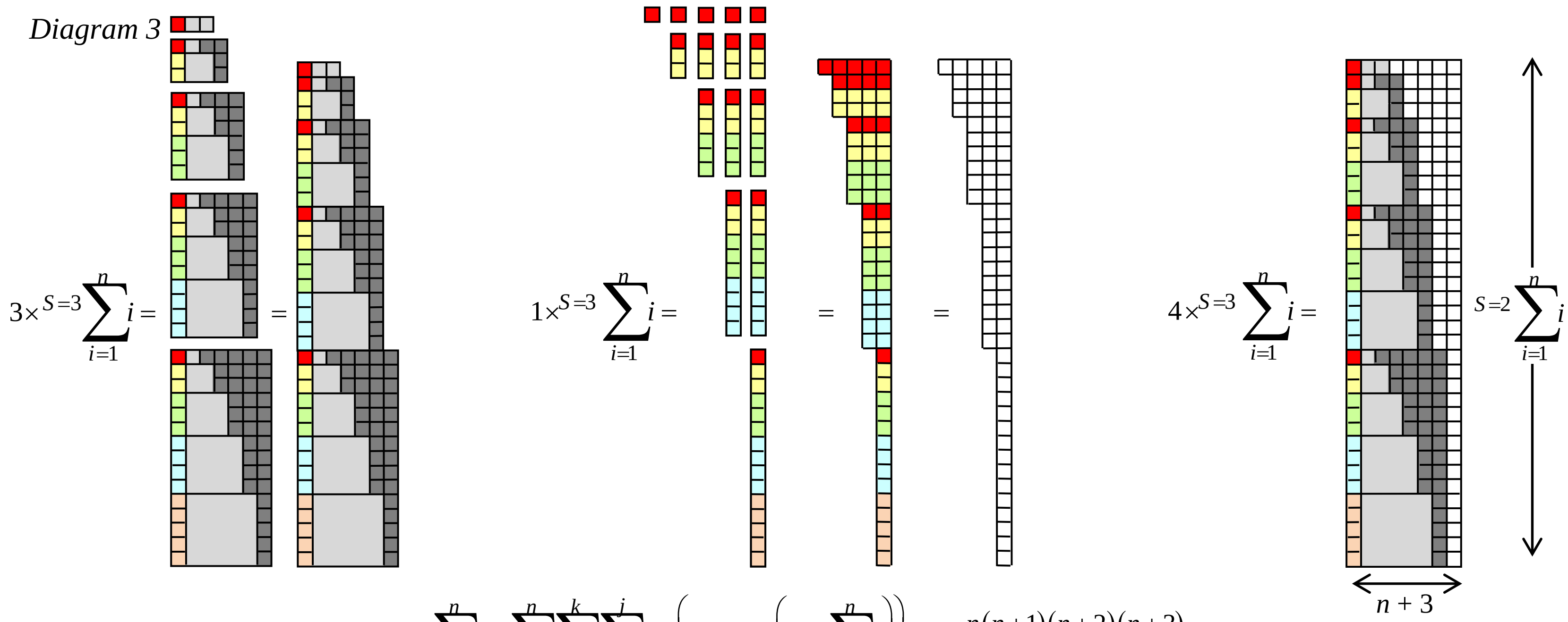
Diagram 2



from: Jan van Maanen, 2019

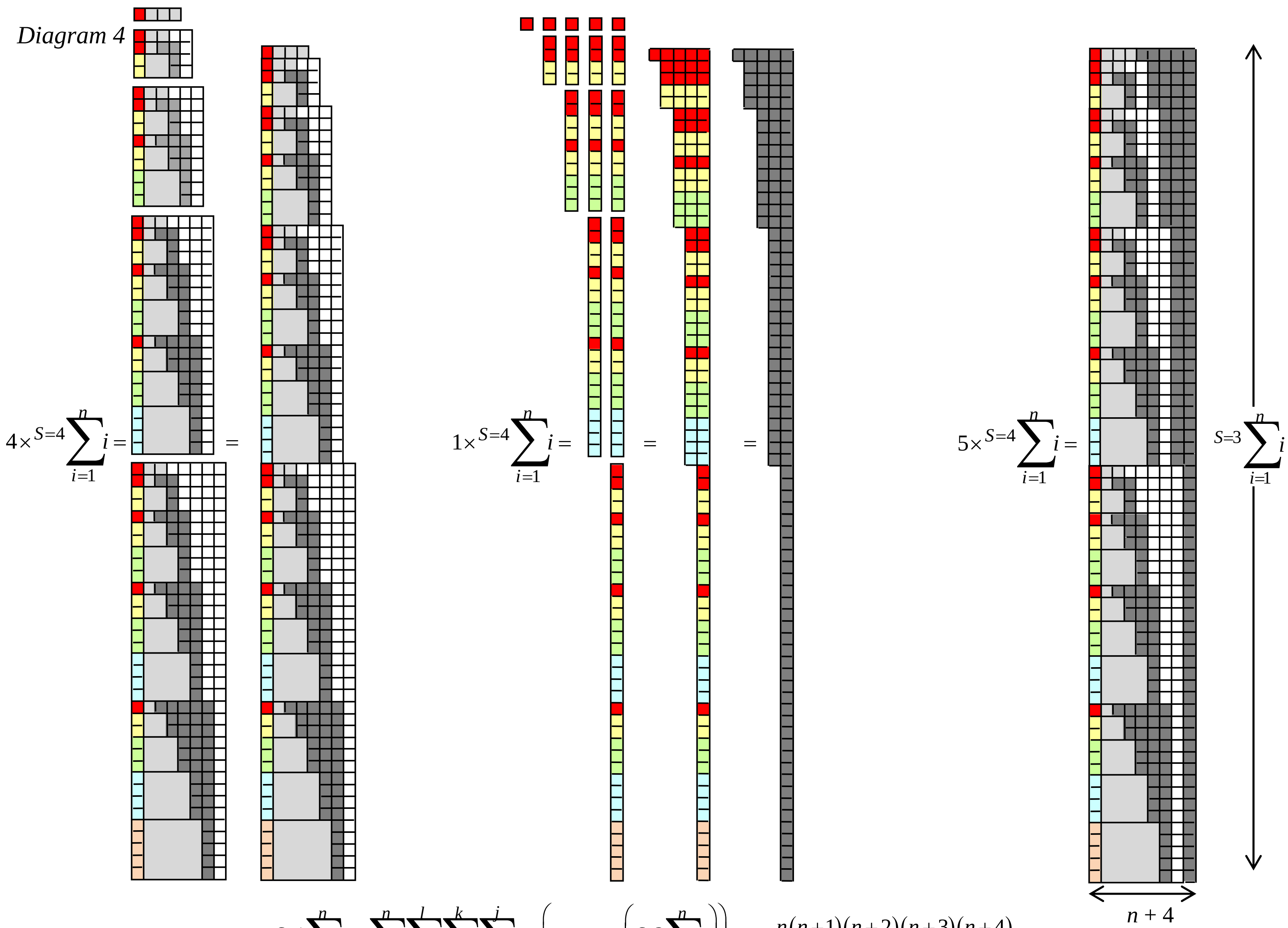
$$^{S=2}\sum_{i=1}^{n} i = \sum_{j=1}^{n}\sum_{i=1}^{j} i = \left((n+2)\times\left({}^{S=1}\sum_{i=1}^{n} i\right)\right)\div 3 = \frac{n(n+1)(n+2)}{2\times3}$$

Diagram 3



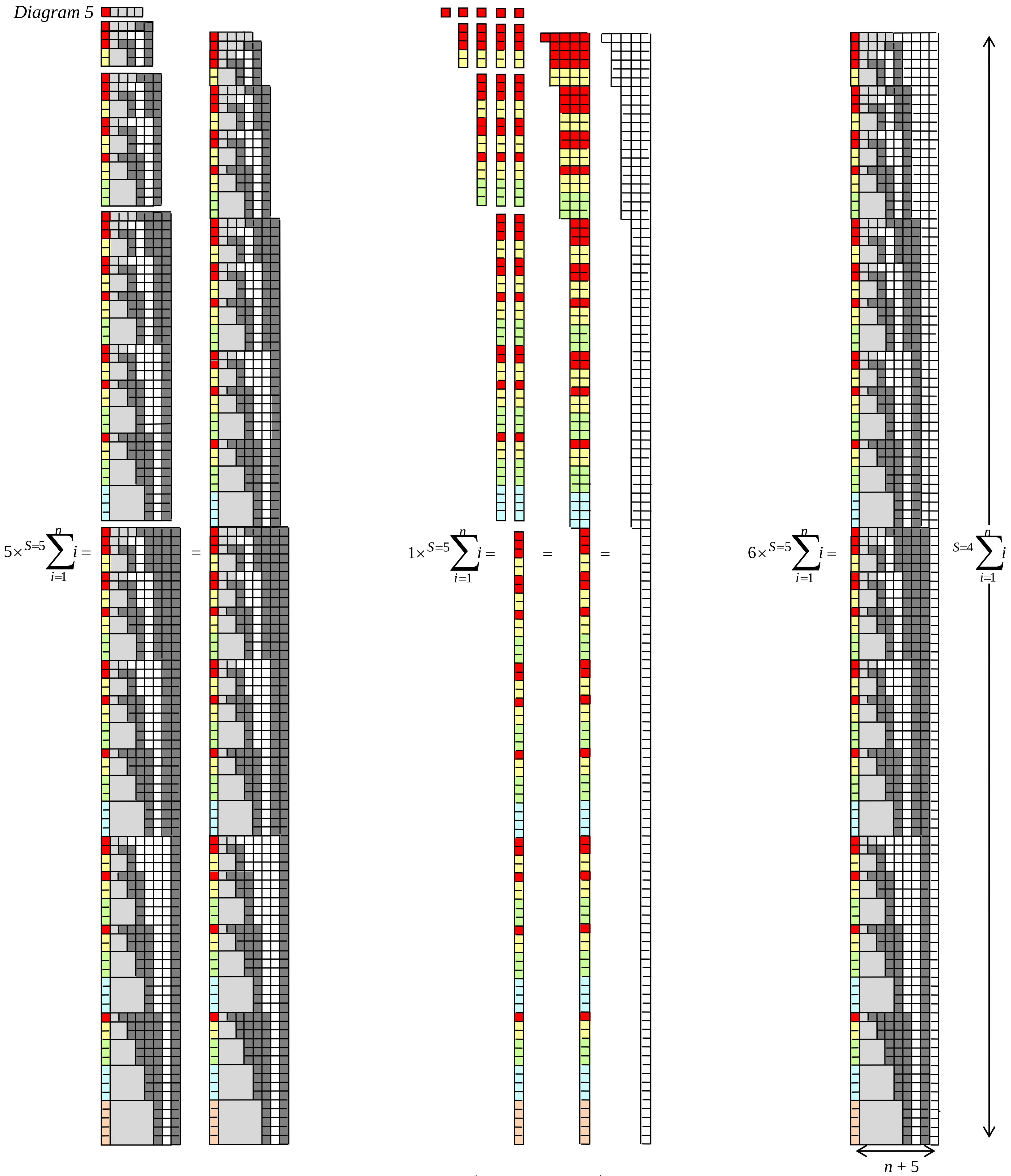
$$^{S=3}\sum_{i=1}^{n} i = \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} i = \left((n+3)\times\left({}^{S=2}\sum_{i=1}^{n} i\right)\right)\div 4 = \frac{n(n+1)(n+2)(n+3)}{2\times3\times4}$$

Diagram 4



$$^{S=4}\sum_{i=1}^{n} i = \sum_{l=1}^{n}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i = \left((n+4)\times\left({}^{S=3}\sum_{i=1}^{n} i\right)\right)\div 5 = \frac{n(n+1)(n+2)(n+3)(n+4)}{2\times3\times4\times5}$$

Diagram 5



$$ {}^{S=5}\sum_{i=1}^{n} i = \sum_{m=1}^{n}\sum_{l=1}^{m}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i = \left((n+5)\times\left({}^{S=4}\sum_{i=1}^{n} i\right)\right)\div 6 = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{2\times3\times4\times5\times6} $$

$$ {}^{S=1}\sum_{i=1}^{n} i = \frac{n(n+1)}{2} $$

$$ {}^{S=2}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)}{2\times3} $$

$$ {}^{S=3}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)(n+3)}{2\times3\times4} $$

$$ {}^{S=4}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)(n+3)(n+4)}{2\times3\times4\times5} $$

$$ {}^{S=5}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{2\times3\times4\times5\times6} $$

by induction

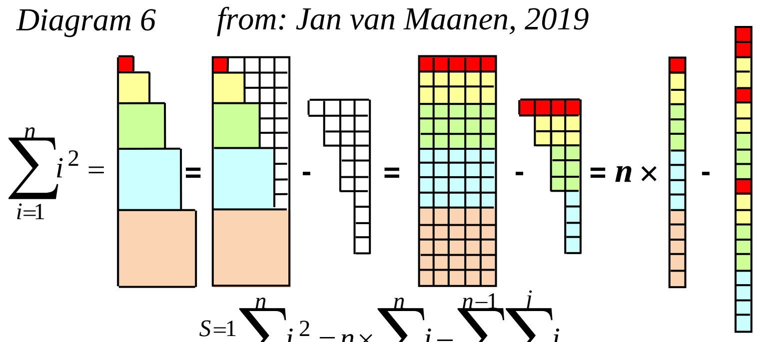
$$ {}^{S}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)....(n+S)}{(S+1)!} $$

or

$$ {}^{S}\sum_{i=1}^{n} i = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} $$

Sums of squares

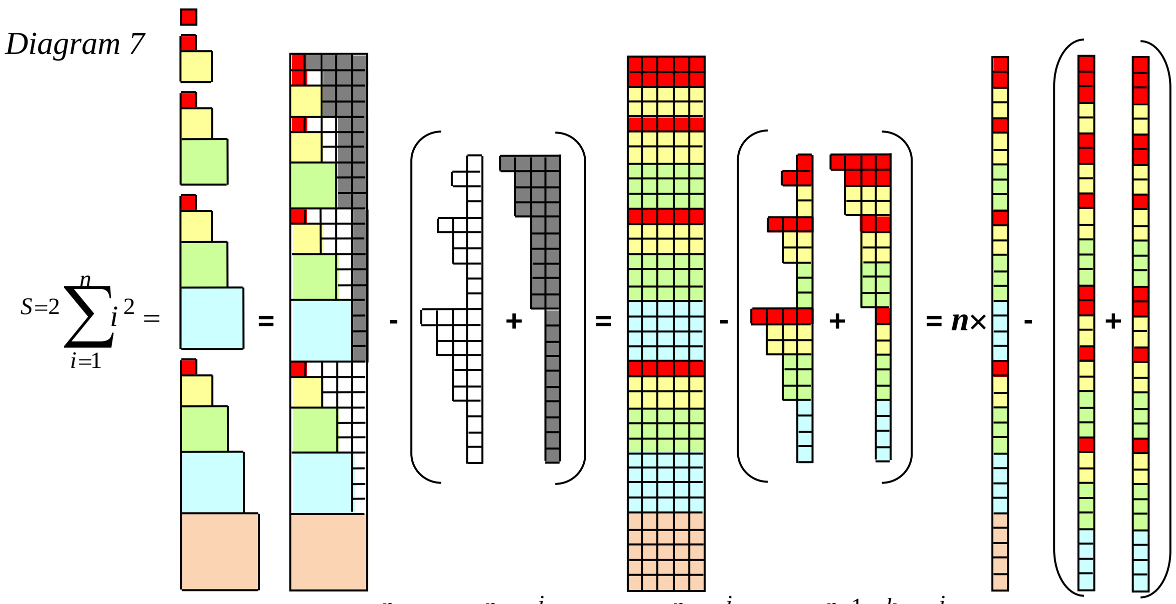
Diagram 6 *from: Jan van Maanen, 2019*



$$ {}^{S=1}\sum_{i=1}^{n} i^2 = n\times\sum_{i=1}^{n} i - \sum_{j=1}^{n-1}\sum_{i=1}^{j} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=1}\sum_{i=1}^{n} i\right) - \left({}^{S=2}\sum_{i=1}^{n-1} i\right) $$

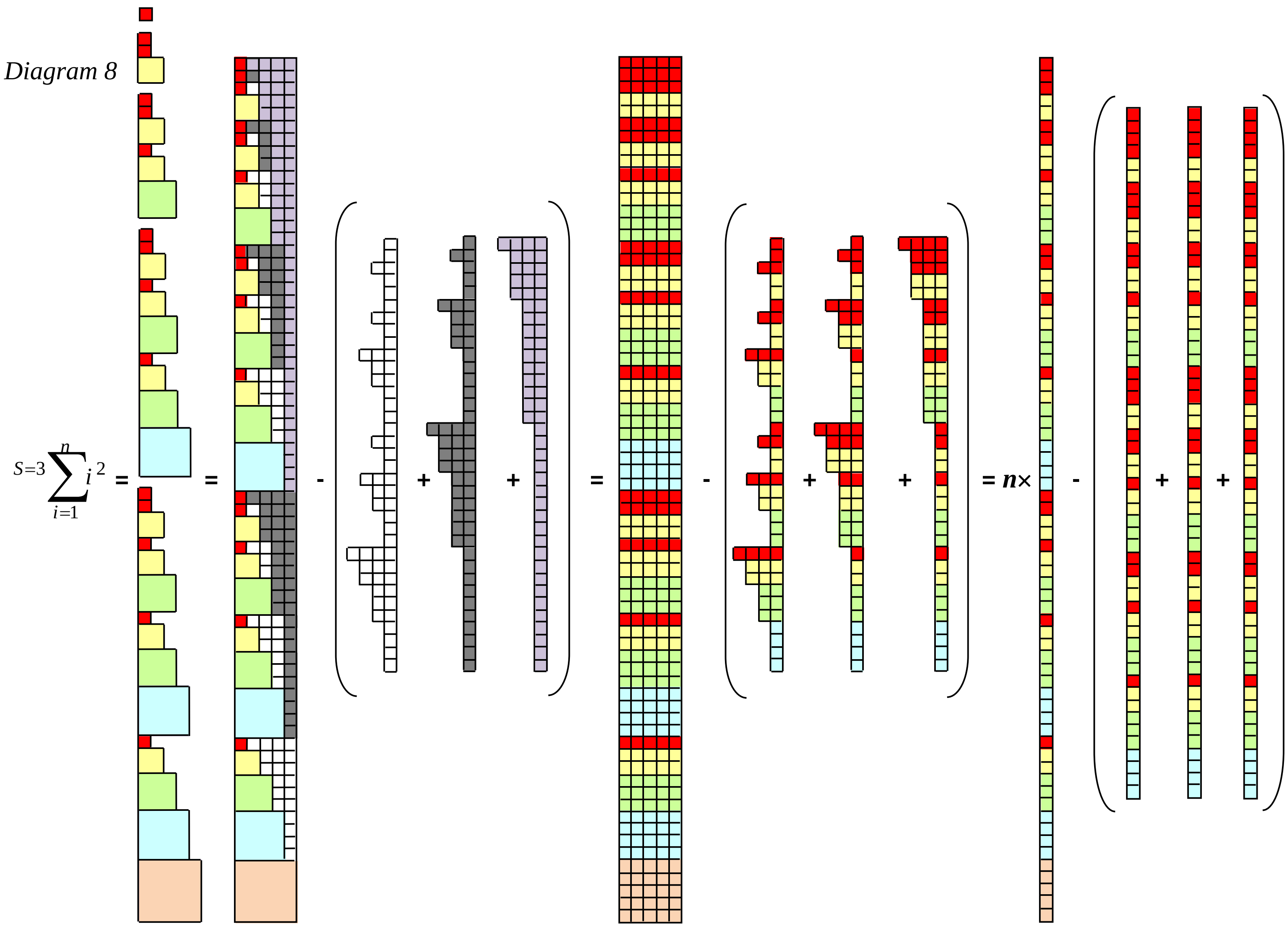
Diagram 7



$$ {}^{S=2}\sum_{i=1}^{n} i^2 = \sum_{j=1}^{n}\sum_{i=1}^{j} i^2 = n\times\sum_{j=1}^{n}\sum_{i=1}^{j} i - 2\times\sum_{k=1}^{n-1}\sum_{j=1}^{k}\sum_{i=1}^{j} i $$

$$ {}^{S=2}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=2}\sum_{i=1}^{n} i\right) - 2\times\left({}^{S=3}\sum_{i=1}^{n-1} i\right) $$

Diagram 8



$$ {}^{S=3}\sum_{i=1}^{n} i^2 = \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} i^2 = n\times\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} i - 3\times\sum_{l=1}^{n-1}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i $$

$$ {}^{S=3}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=3}\sum_{i=1}^{n} i\right) - 3\times\left({}^{S=4}\sum_{i=1}^{n-1} i\right) $$

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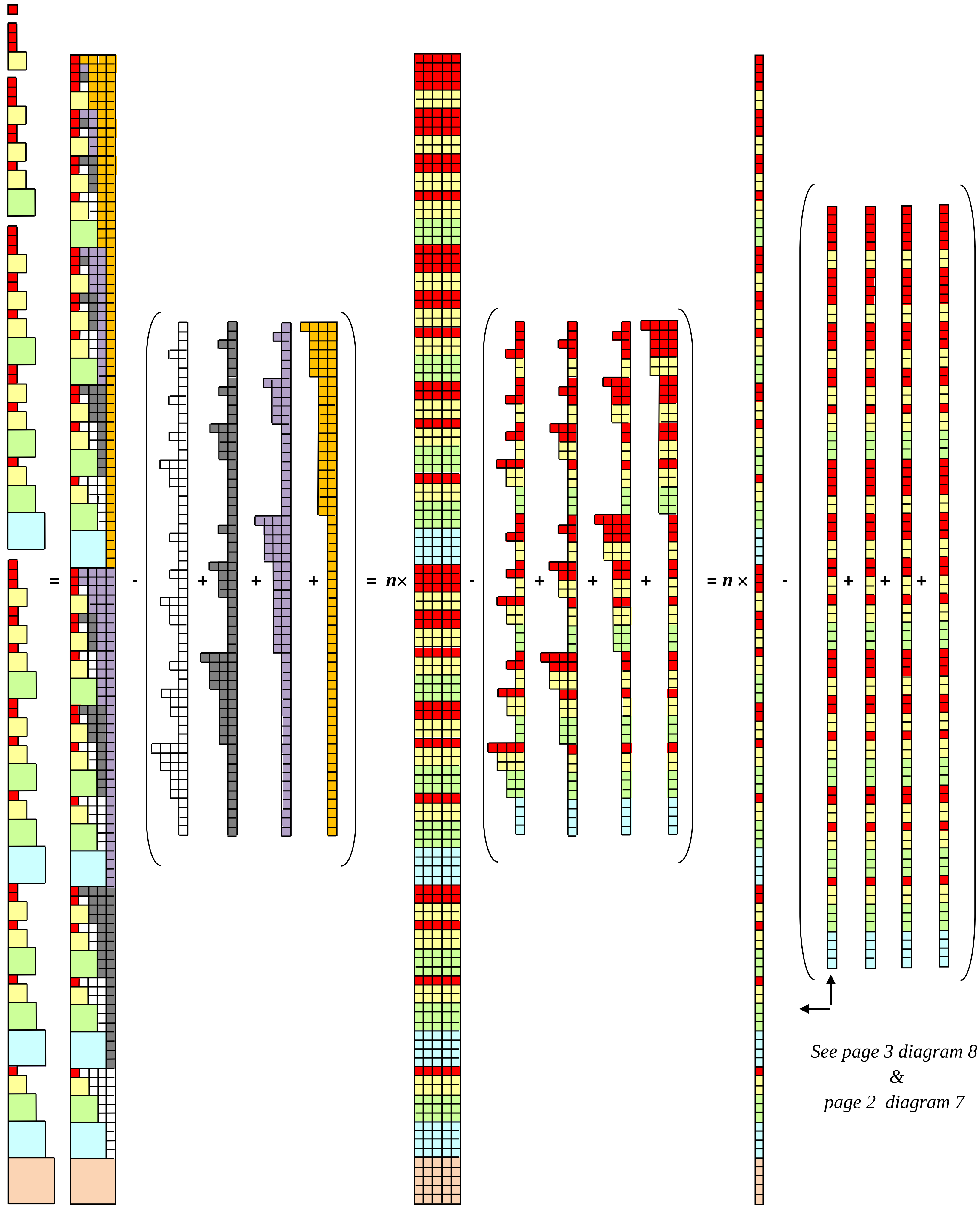
Diagram 9

$${}^{S=4}\sum_{i=1}^{n} i^2 =$$



$${}^{S=4}\sum_{i=1}^{n} i^2 = \sum_{l=1}^{n}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i^2 = n\times\sum_{l=1}^{n}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i - 4\times\sum_{m=1}^{n-1}\sum_{l=1}^{m}\sum_{k=1}^{l}\sum_{j=1}^{k}\sum_{i=1}^{j} i$$

$${}^{S=4}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=4}\sum_{i=1}^{n} i\right) - 4\times\left({}^{S=5}\sum_{i=1}^{n-1} i\right)$$

${}^{S=1}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=1}\sum_{i=1}^{n} i\right) - \left({}^{S=2}\sum_{i=1}^{n-1} i\right)$	${}^{S=2}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=2}\sum_{i=1}^{n} i\right) - 2\times\left({}^{S=3}\sum_{i=1}^{n-1} i\right)$	${}^{S=3}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=3}\sum_{i=1}^{n} i\right) - 3\times\left({}^{S=4}\sum_{i=1}^{n-1} i\right)$	${}^{S=4}\sum_{i=1}^{n} i^2 = n\times\left({}^{S=4}\sum_{i=1}^{n} i\right) - 4\times\left({}^{S=5}\sum_{i=1}^{n-1} i\right)$

by induction

$${}^{S}\sum_{i=1}^{n} i^2 = n\times\left({}^{S}\sum_{i=1}^{n} i\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i\right)$$

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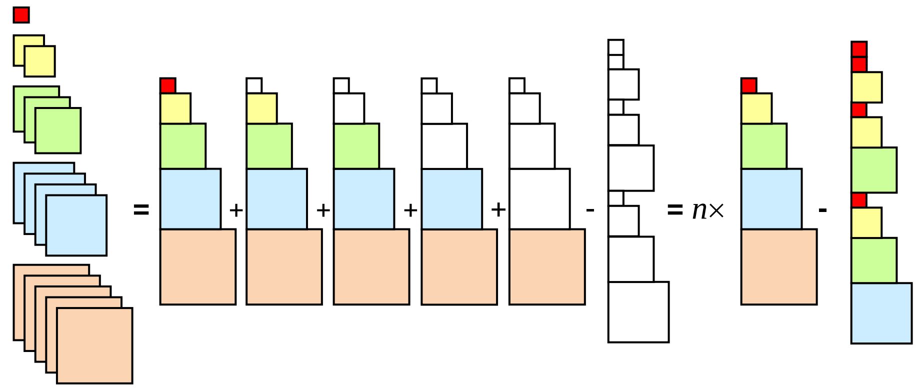
Sums of cubes

Diagram 10

$$\sum_{i=1}^{n}{}_{S=1}\, i^3 =$$

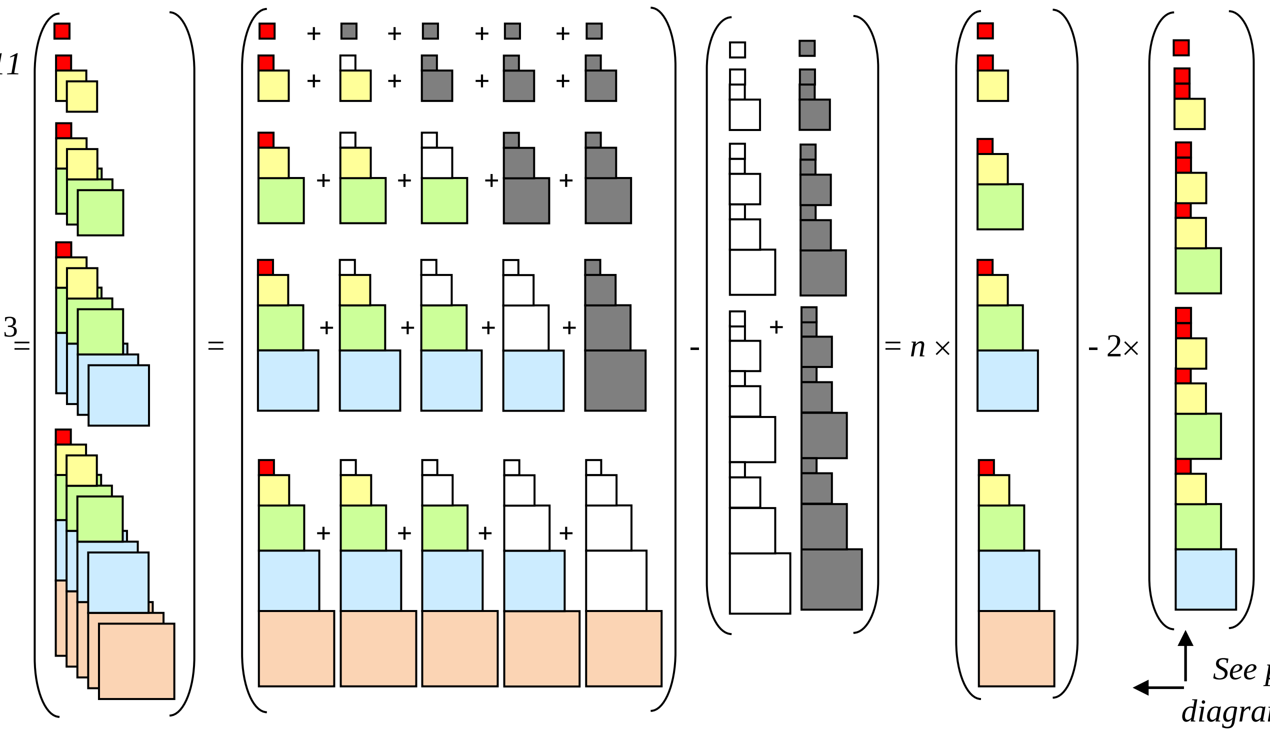


See page 4 diagrams 6 & 7

$$\sum_{i=1}^{n}{}_{S=1}\, i^3 = n\times\left(\sum_{i=1}^{n}{}_{S=1}\, i^2\right)-\left(\sum_{i=1}^{n-1}{}_{S=2}\, i^2\right)$$

Diagram 11

$$\sum_{i=1}^{n}{}_{S=2}\, i^3 =$$

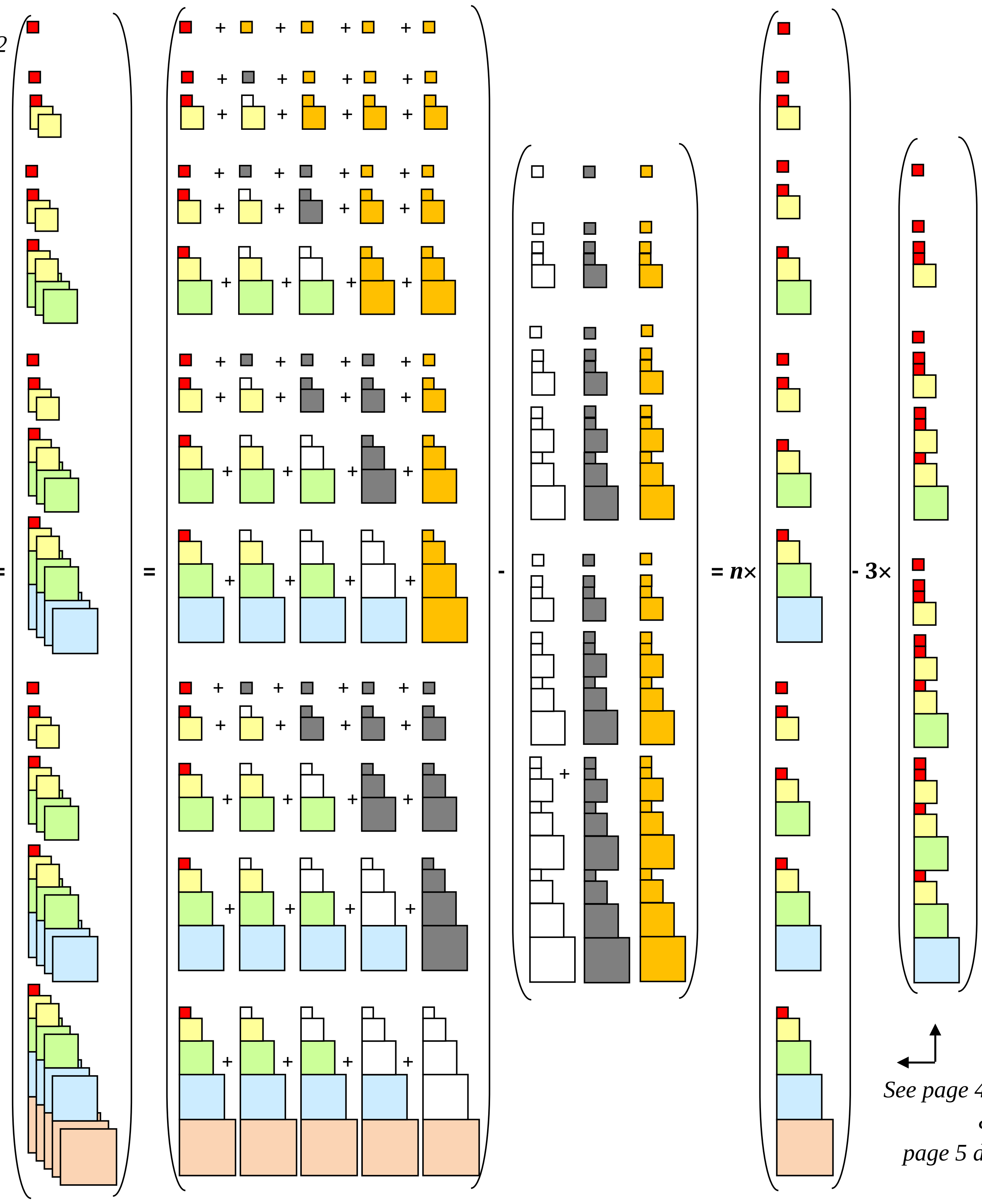


See page 4 diagrams 7 & 8

$$\sum_{i=1}^{n}{}_{S=2}\, i^3 = n\times\left(\sum_{i=1}^{n}{}_{S=2}\, i^2\right)-2\times\left(\sum_{i=1}^{n-1}{}_{S=3}\, i^2\right)$$

Diagram 12

$$\sum_{i=1}^{n}{}_{S=3}\, i^3 =$$



See page 4 diagram 8 & page 5 diagram 9

$$\sum_{i=1}^{n}{}_{S=3}\, i^3 = n\times\left(\sum_{i=1}^{n}{}_{S=3}\, i^2\right)-3\times\left(\sum_{i=1}^{n-1}{}_{S=4}\, i^2\right)$$

by induction

$$\sum_{i=1}^{n}{}_{S}\, i^3 = n\times\left(\sum_{i=1}^{n}{}_{S}\, i^2\right)-S\times\left(\sum_{i=1}^{n-1}{}_{S+1}\, i^2\right)$$

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Intermezzo

These are the geometric representations of the sums of the cubes which will be used to derive the sums of the fourth power:

Diagram 13[1]

$S = 1\sum_{i=1}^{n} i^3 =$

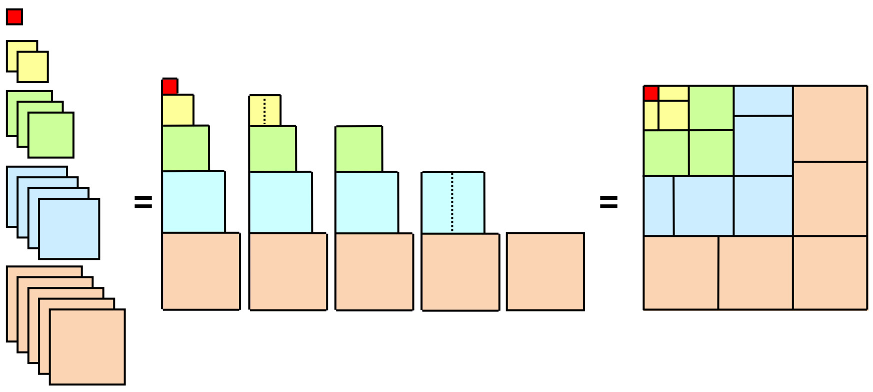
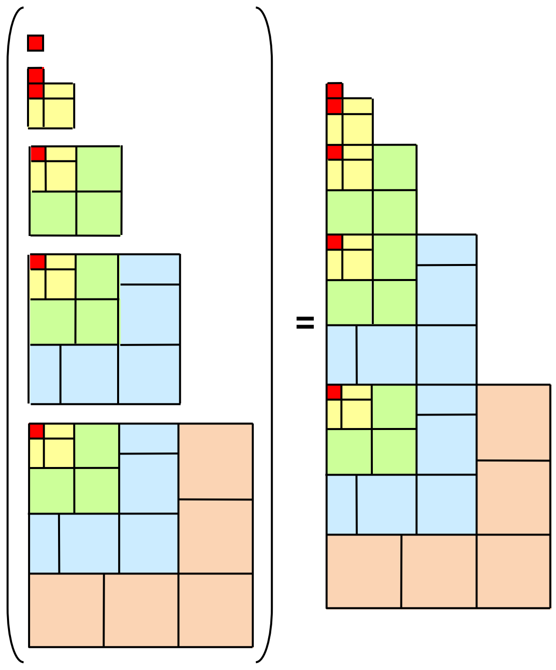


Diagram 14

$S = 2\sum_{i=1}^{n} i^3 =$



$S = 2\sum_{i=1}^{n-1} i^3 =$

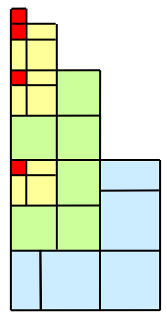
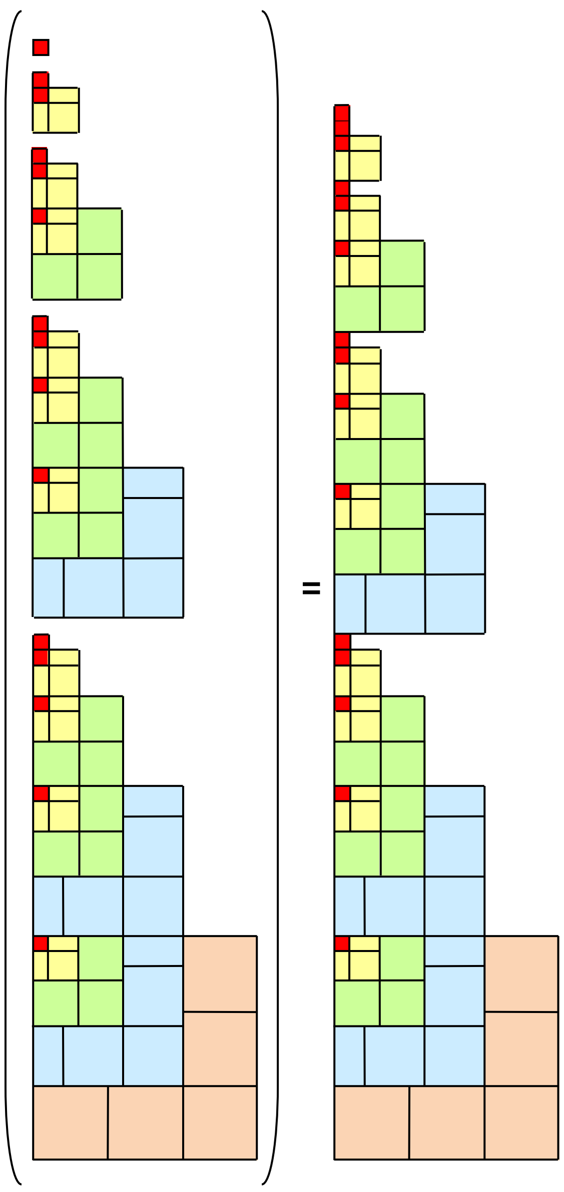
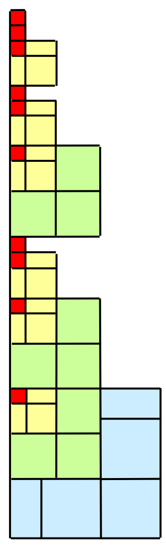


Diagram 15

$S = 3\sum_{i=1}^{n} i^3 =$



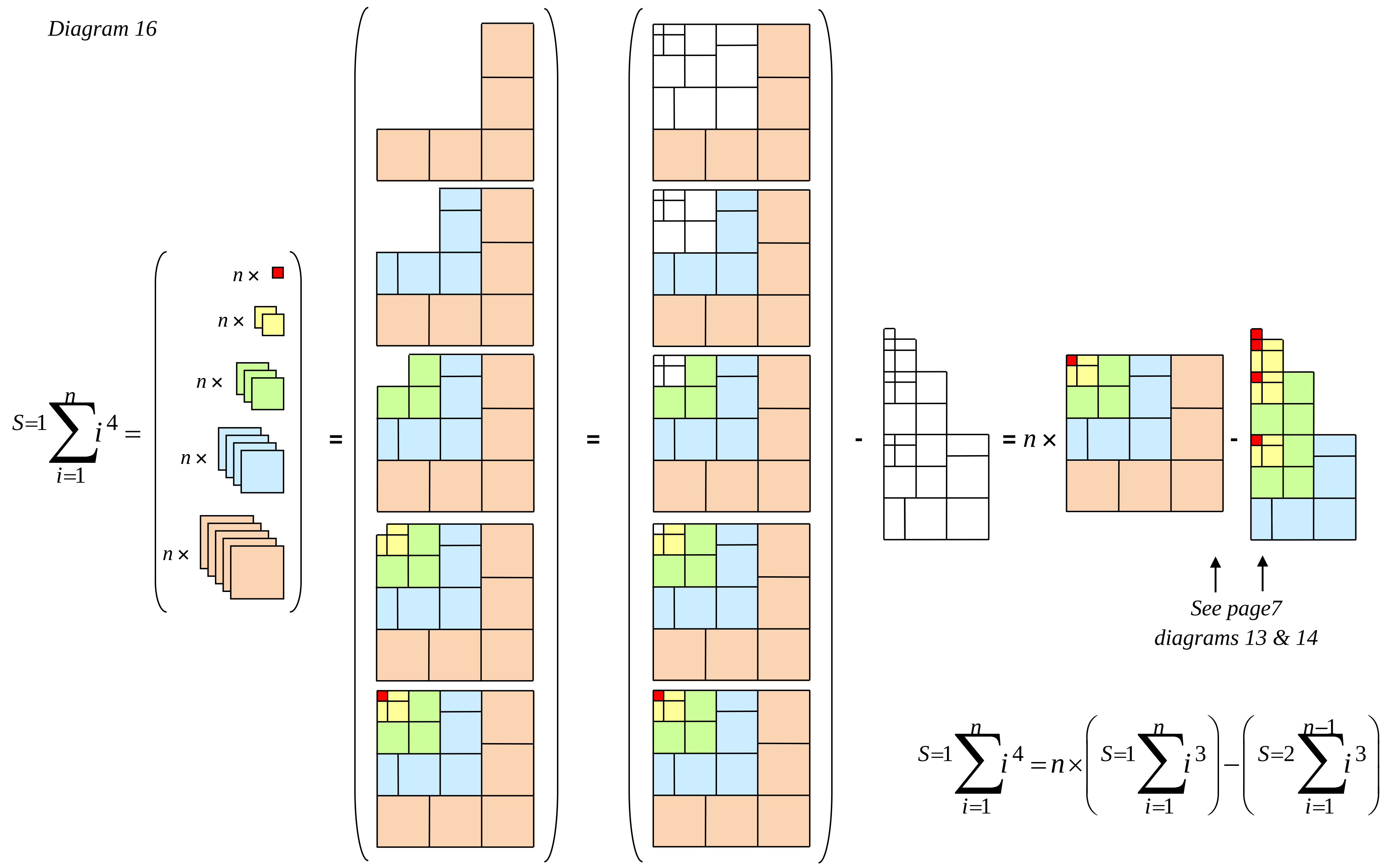
$S = 3\sum_{i=1}^{n-1} i^3 =$



[1] *Diagram 13* from Al-Karaji (935 ~ 1029AD)

Sums of the fourth power

Diagram 16



$$\sum_{S=1}^{n} i^4 = n \times \left(\sum_{S=1}^{n} i^3 \right) - \left(\sum_{S=2}^{n-1} i^3 \right)$$

Diagram 17

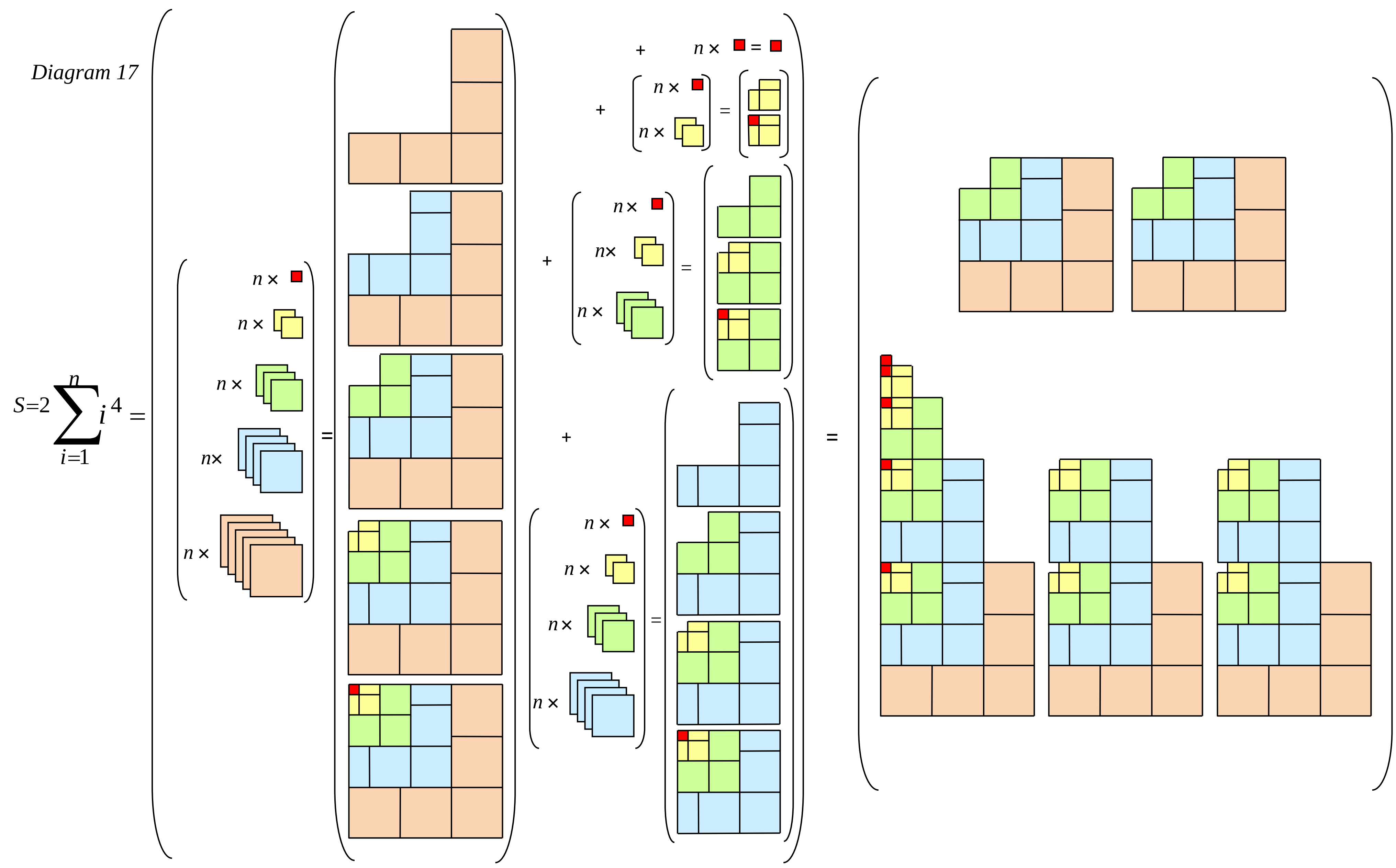


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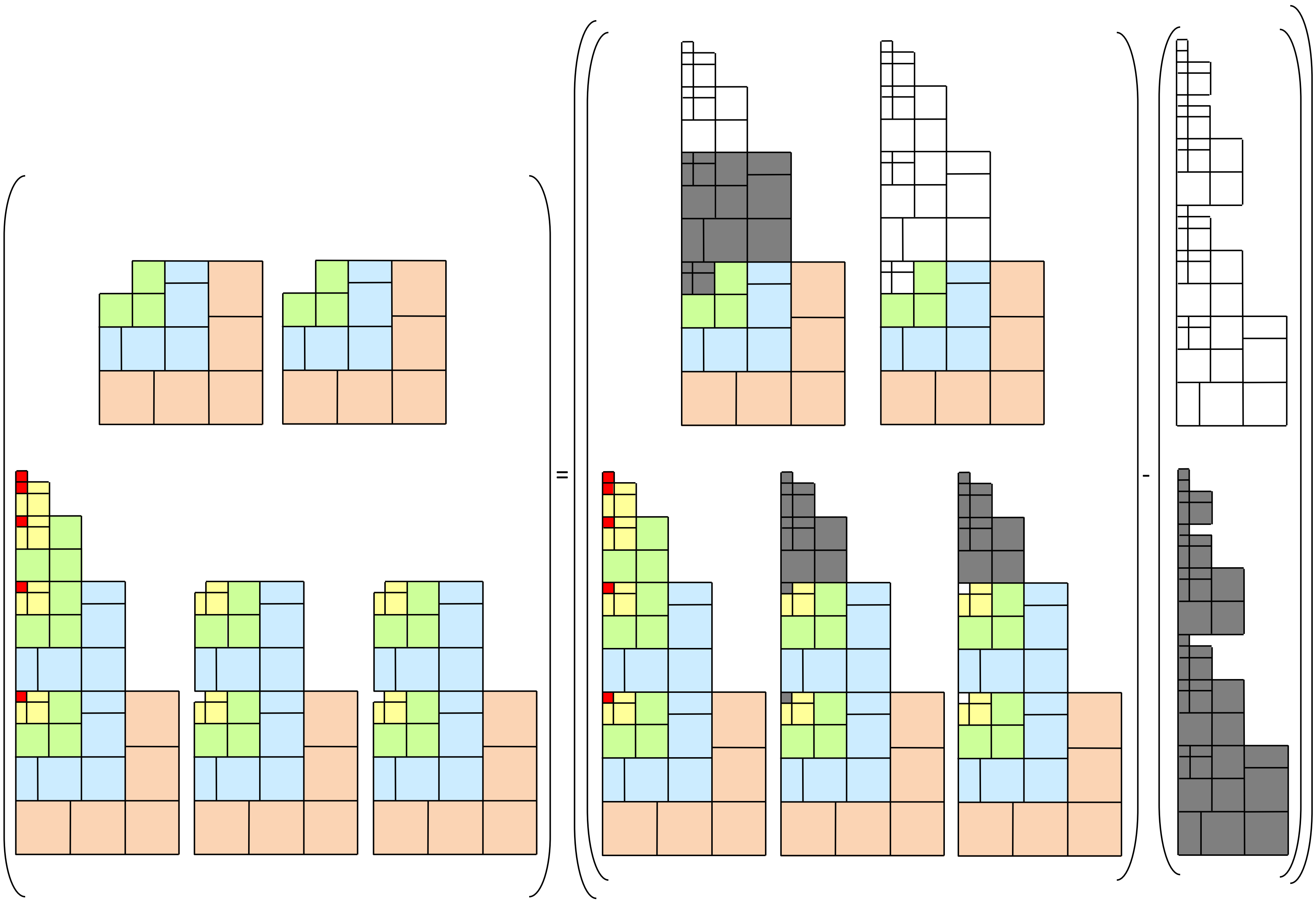
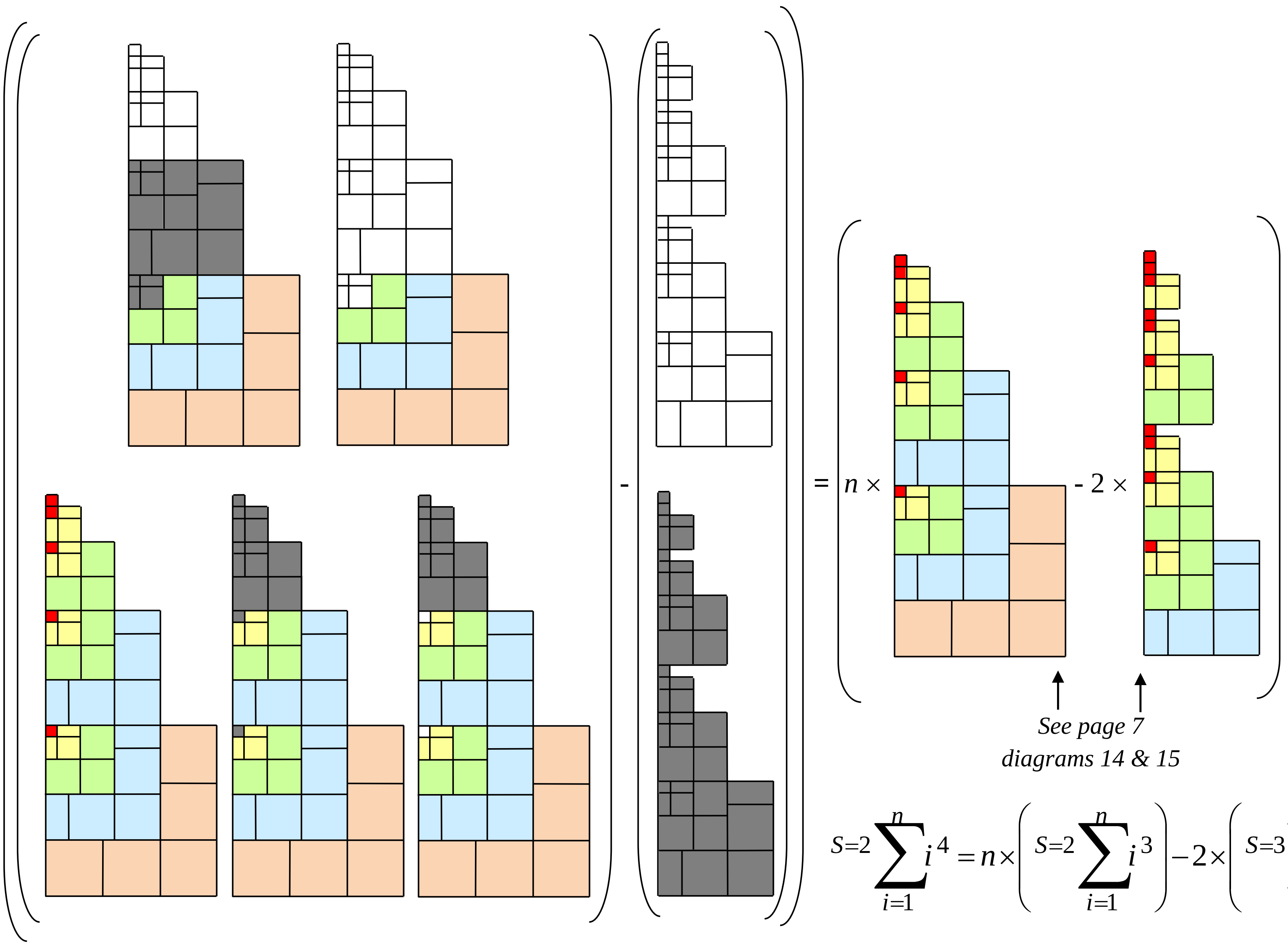
Diagram 17
continued

$$S{=}2\sum_{i=1}^{n} i^4 =$$



$$S{=}2\sum_{i=1}^{n} i^4 =$$



See page 7
diagrams 14 & 15

$$S{=}2\sum_{i=1}^{n} i^4 = n\times\left(S{=}2\sum_{i=1}^{n} i^3\right) - 2\times\left(S{=}3\sum_{i=1}^{n-1} i^3\right)$$

$$\boxed{S{=}1\sum_{i=1}^{n} i^4 = n\times\left(S{=}1\sum_{i=1}^{n} i^3\right) - \left(S{=}2\sum_{i=1}^{n-1} i^3\right)}$$

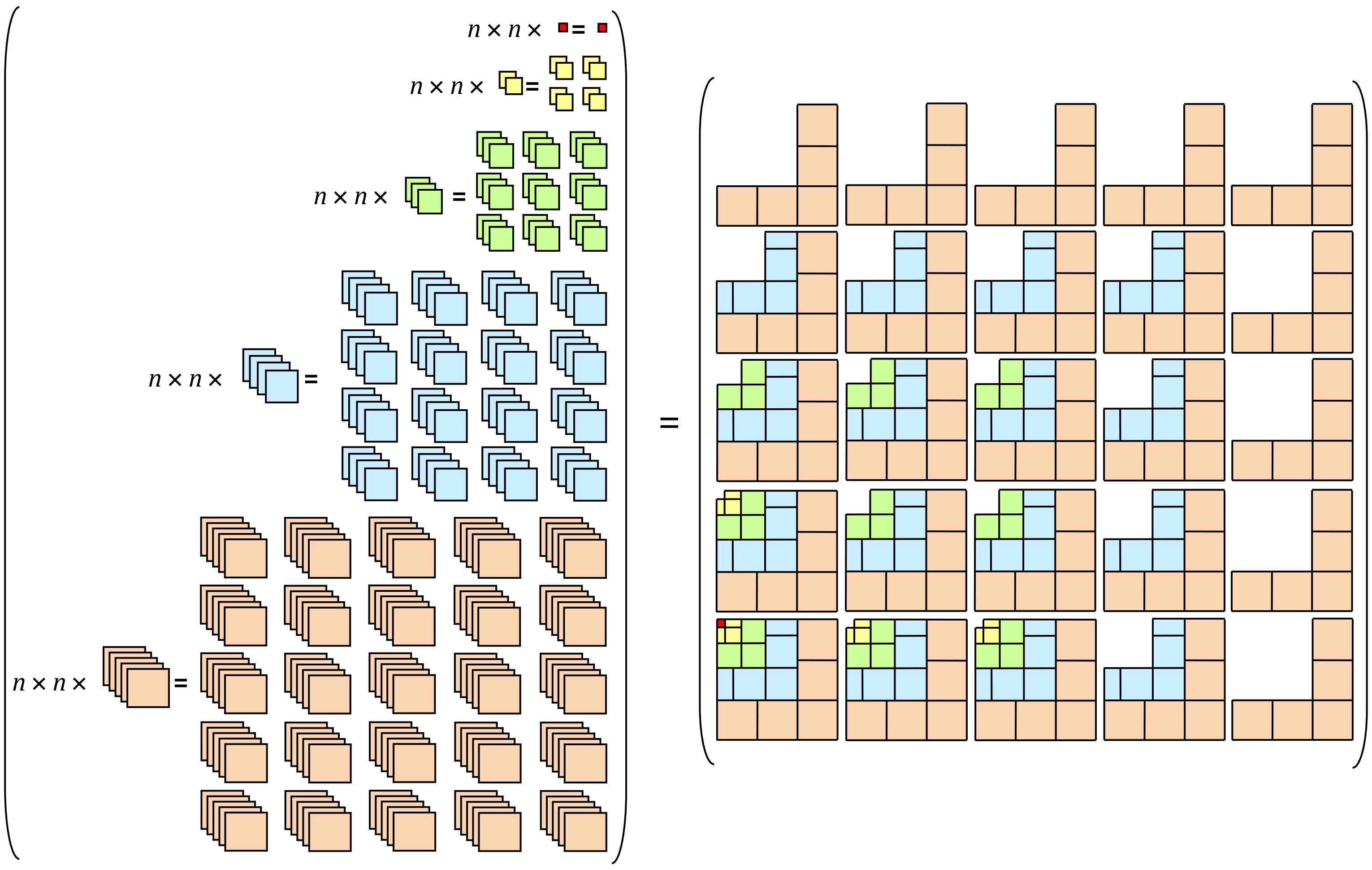
&

$$\boxed{S{=}2\sum_{i=1}^{n} i^4 = n\times\left(S{=}2\sum_{i=1}^{n} i^3\right) - 2\times\left(S{=}3\sum_{i=1}^{n-1} i^3\right)}$$

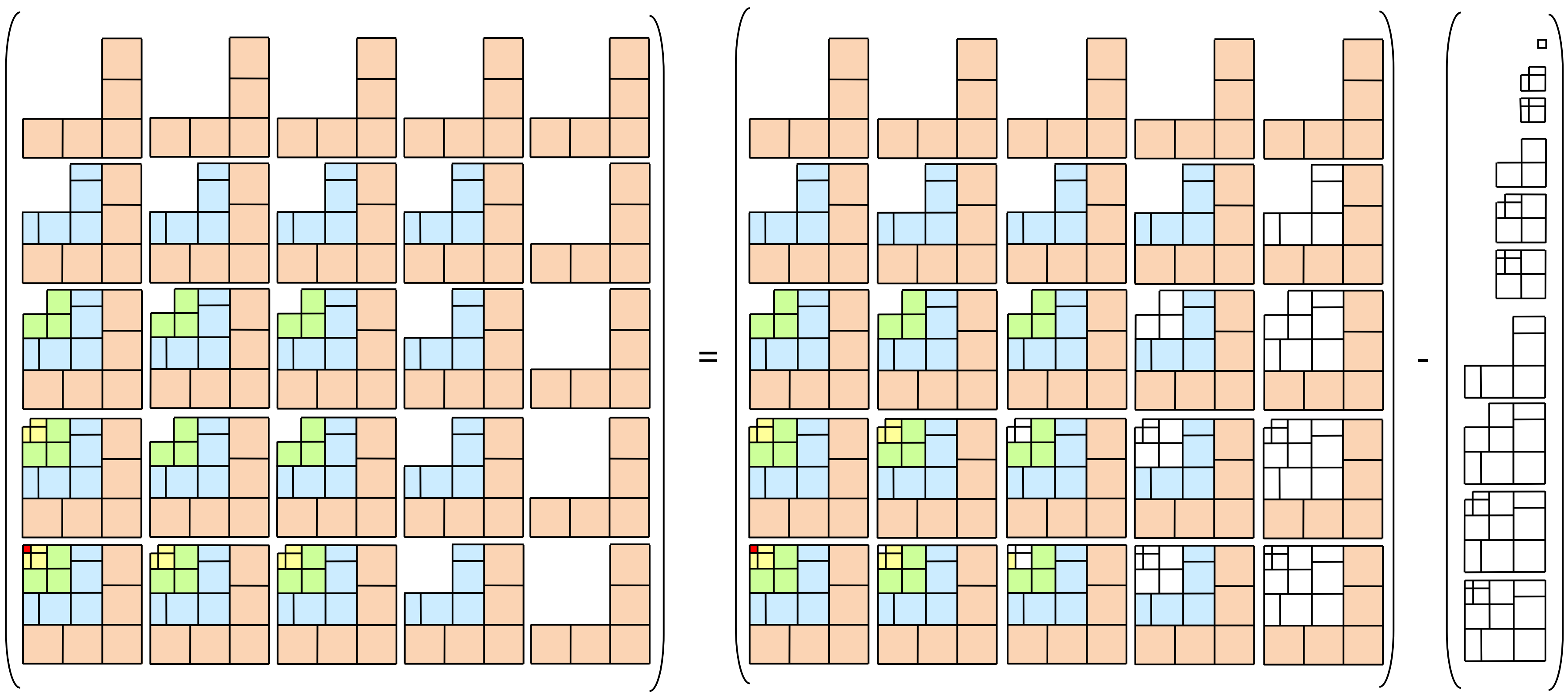
The sum of the fifth power

Diagram 18

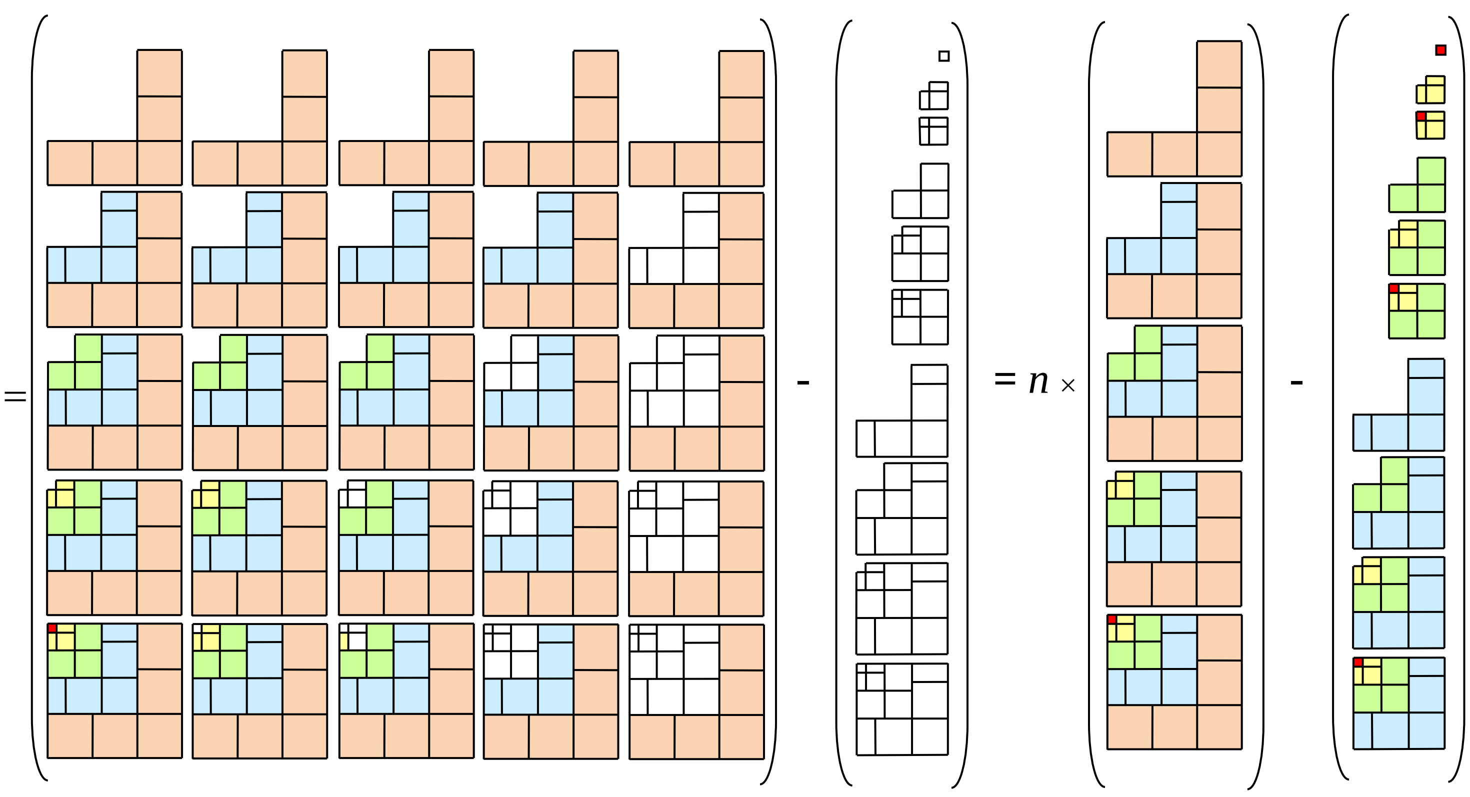
$$S\!=\!1\sum_{i=1}^{n} i^{5} =$$



$$S\!=\!1\sum_{i=1}^{n} i^{5} =$$



$$S\!=\!1\sum_{i=1}^{n} i^{5} =$$



$$S\!=\!1\sum_{i=1}^{n} i^{5} = n \times \left(S\!=\!1\sum_{i=1}^{n} i^{4} \right) - \left(S\!=\!2\sum_{i=1}^{n-1} i^{4} \right)$$

See page 8 diagrams 16 & 17

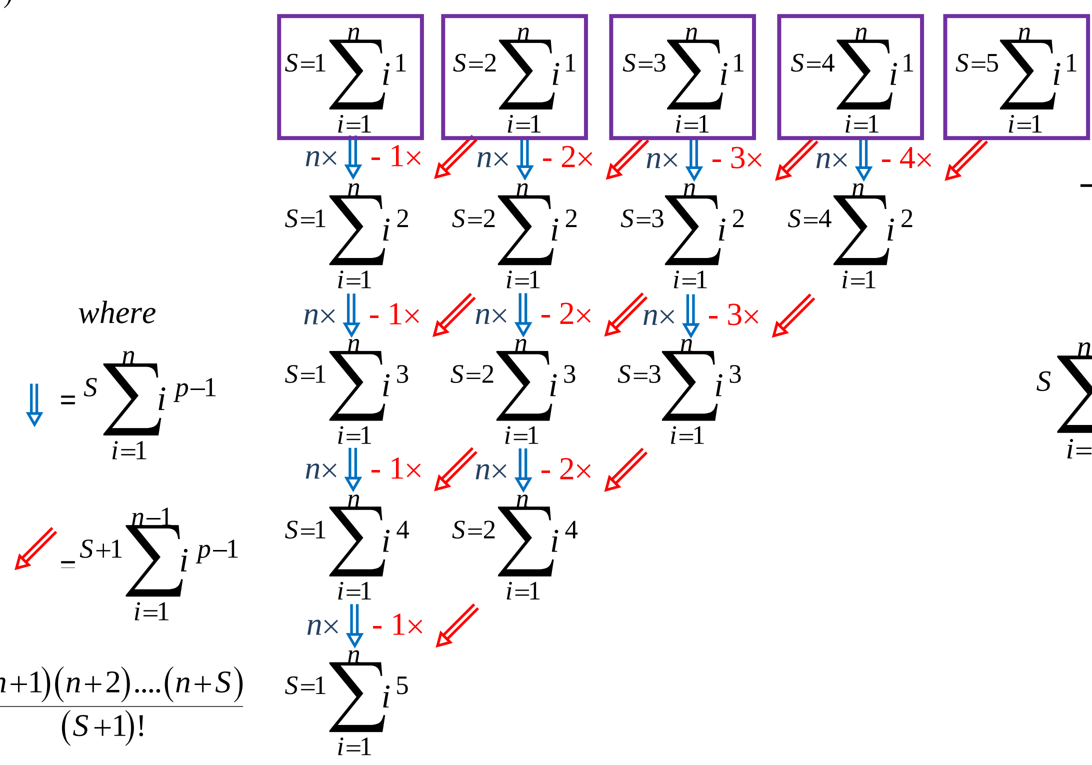
Algebraic induction

Here is a summary of the geometric proofs on pages 2 to 10:

from pages 2 & 3 *Diagram. 19* *from pages 4 to 10*

$${}^{S}\sum_{i=1}^{n} i = \frac{(n+S)!}{(n-1)!(S+1)!}$$

for all powers >1



& where $${}^{S}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)....(n+S)}{(S+1)!}$$

⟶ *by induction*

for all $p>1$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right)$$

A comparison with Alhazen’s formula

(Abu Ali al-Hasan ibn –Hasan ibn Al-Haytham (c. 965 – 1040 AD), also known in the West as Alhazen)

It is possible to derive my formula from Alhazen’s formula,
but it only then describes the situation for when $S = 1$:

Alhazen’s formula:

$$(n+1)\sum_{i=1}^{n} i^{p} = \sum_{i=1}^{n} i^{p+1} + \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p}$$

$$\sum_{i=1}^{n} i^{p+1} = (n+1)\sum_{i=1}^{n} i^{p} - \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p}$$

$$\sum_{i=1}^{n} i^{p} = (n+1)\sum_{i=1}^{n} i^{p-1} - \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p-1}$$

$$\sum_{i=1}^{n} i^{p} = n\sum_{i=1}^{n} i^{p-1} + \sum_{i=1}^{n} i^{p-1} - \left(\sum_{i=1}^{n} i^{p-1} + \sum_{i=1}^{n-1} i^{p-1} + \sum_{i=1}^{n-2} i^{p-1} + ... + \sum_{i=1}^{1} i^{p-1}\right)$$

$$\sum_{i=1}^{n} i^{p} = n\sum_{i=1}^{n} i^{p-1} - \left(\sum_{i=1}^{n-1} i^{p-1} + \sum_{i=1}^{n-2} i^{p-1} + ... + \sum_{i=1}^{1} i^{p-1}\right)$$

$$\sum_{i=1}^{n} i^{p} = n\sum_{i=1}^{n} i^{p-1} - \sum_{k=1}^{n-1}\sum_{i=1}^{k} i^{p-1}$$

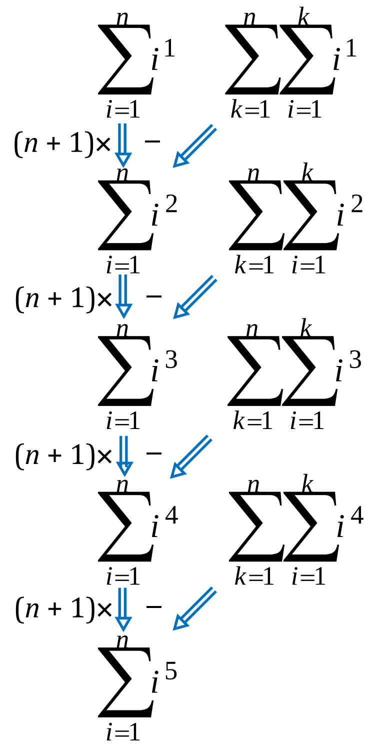
when $S=1$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right) = \sum_{i=1}^{n} i^{p} = n\times\sum_{i=1}^{n} i^{p-1} - \sum_{k=1}^{n-1}\sum_{i=1}^{k} i^{p-1}$$

Diagram. 20 Alhazen’s formula

$$(n+1)\sum_{i=1}^{n} i^{p} = \sum_{i=1}^{n} i^{p+1} + \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p}$$

$$\sum_{i=1}^{n} i^{p+1} = (n+1)\sum_{i=1}^{n} i^{p} - \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p}$$



A comparison of *diagrams 19* & *20* shows that Alhazen’s formula only derives the sum of the power $\boldsymbol{p}$ from the sum of the power $\boldsymbol{p} - 1$ minus the sum of the sums of the power $\boldsymbol{p} - 1$, but not how these sums of sums may themselves be derived. Whereas my formula shows that the derivations of all the sums of powers may ultimate be made geometrically from the sums of the sums of the integers, which themselves may all be derived geometrically from the sum of the integers. Clearly, my formula provides a wider understanding of the geometric causality behind Alhazen’s formula.

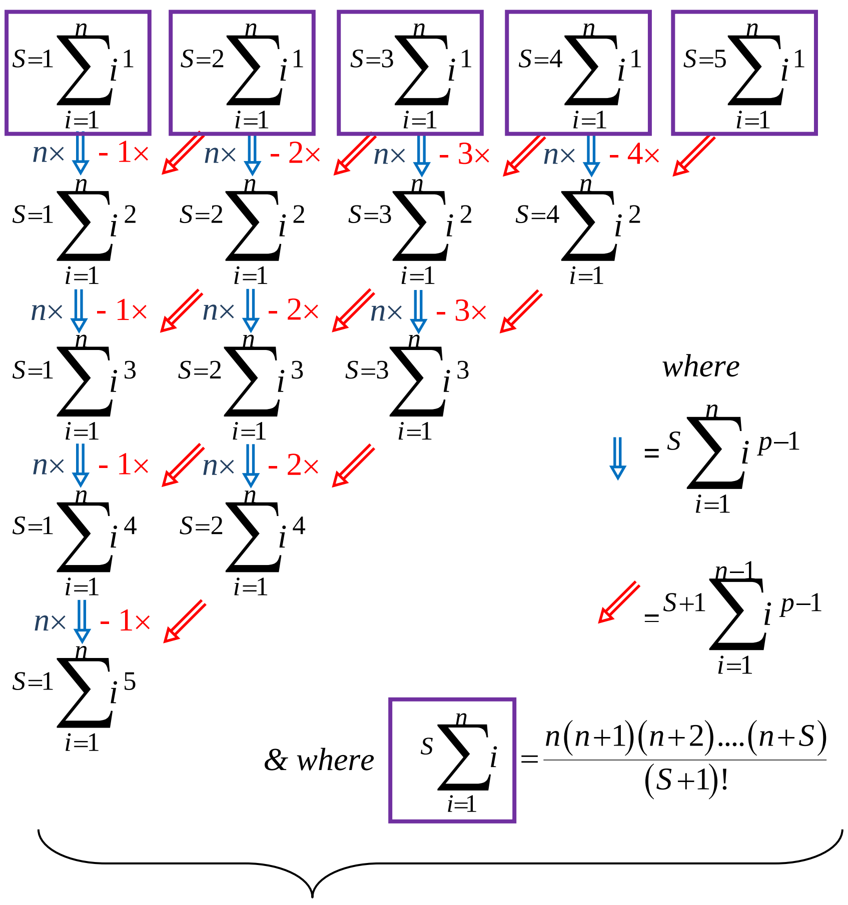
According to Katz[1], Alhazen first derived his formula for $\boldsymbol{p} = 4$ by using a technique of arithmetical induction – whereas all the derivations in my *diagram 19* were made by the more pedestrian geometric method of proof by congruence.

[1] pages 255 – 259, Victor J. Katz, *A History of Mathematics, An Introduction*. 2nd Ed. 1998. Addison-Wesley

Completing Alhazen's triangle

My formula may be used to derive a formula to complete Alhazen's triangle in this way:

Diagram. 19 *for all powers* >1



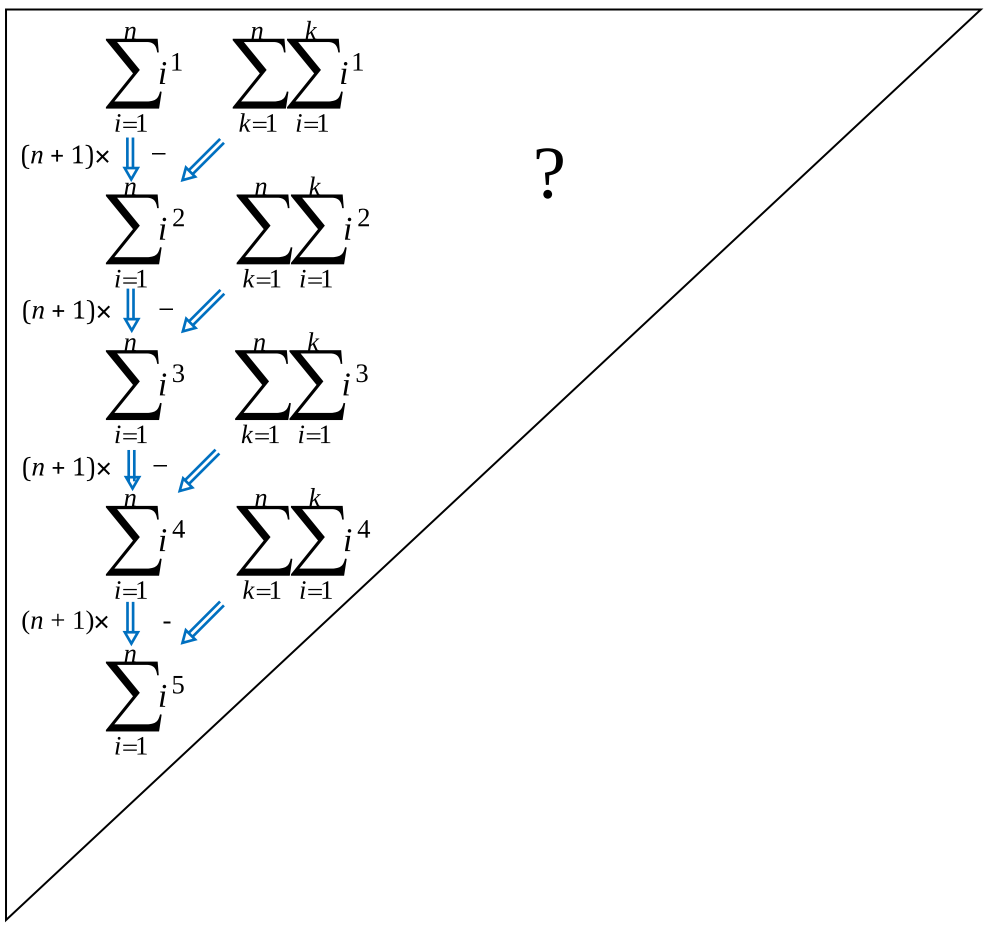
where $\Downarrow = {}^{S}\sum_{i=1}^{n} i^{p-1}$

$\swarrow = {}^{S+1}\sum_{i=1}^{n-1} i^{p-1}$

& where ${}^{S}\sum_{i=1}^{n} i = \dfrac{n(n+1)(n+2)....(n+S)}{(S+1)!}$

Diagram 20 Alhazen's formula:

$$\sum_{i=1}^{n} i^{p+1} = (n+1)\sum_{i=1}^{n} i^{p} - \sum_{k=1}^{n}\sum_{i=1}^{k} i^{p}$$



for all $p>1$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right)$$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) + S\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left(\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) + \left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right)\right)$$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) + S\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left(\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) + \left({}^{S}\sum_{i=1}^{n-1} i^{p-1} + {}^{S}\sum_{i=1}^{n-2} i^{p-1} + ... + {}^{S}\sum_{i=1}^{1} i^{p-1}\right)\right)$$

$${}^{S}\sum_{i=1}^{n} i^{p} = (n+S)\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n} i^{p-1}\right)$$

$${}^{S}\sum_{i=1}^{n} i^{p+1} = (n+S)\times\left({}^{S}\sum_{i=1}^{n} i^{p}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n} i^{p}\right)$$

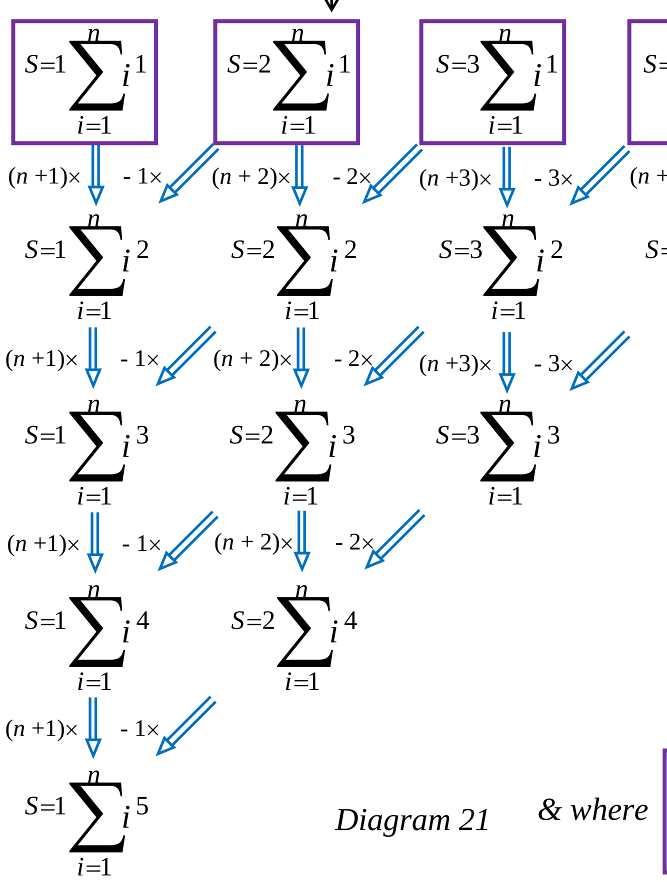


Diagram 21

where $\Downarrow = {}^{S}\sum_{i=1}^{n} i^{p}$

$\swarrow = {}^{S+1}\sum_{i=1}^{n} i^{p}$

& where ${}^{S}\sum_{i=1}^{n} i = \dfrac{(n+S)!}{(n-1)!(S+1)!}$

Diagram 21

Alhazen's completed triangle

$${}^{S}\sum_{i=1}^{n} i^{p+1} = (n+S)\times\left({}^{S}\sum_{i=1}^{n} i^{p}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n} i^{p}\right)$$

It would be possible to draw geometric diagrams to demonstrate the relationships in *diagram 21*, but they would necessarily be larger and more cumbersome than the ones from which I derived my formula.

Discussion

$$^{S}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)....(n+S)}{(S+1)!} \quad or \quad \frac{1}{(S+1)!}\times\frac{(n+S)}{(n-1)!} \quad - \quad \textit{formula } 1$$

from diagram 19

for all $p > 1$

$$^{S}\sum_{i=1}^{n} i^{\,p} = n \times \left({}^{S}\sum_{i=1}^{n} i^{\,p-1} \right) - S \times \left({}^{S+1}\sum_{i=1}^{n-1} i^{\,p-1} \right) \quad - \quad \textit{formula } 2$$

from diagram 21

Alhazen's completed triangle

$$^{S}\sum_{i=1}^{n} i^{\,p+1} = (n+S) \times \left({}^{S}\sum_{i=1}^{n} i^{\,p} \right) - S \times \left({}^{S+1}\sum_{i=1}^{n} i^{\,p} \right) \quad - \quad \textit{formula } 3$$

The geometric diagrams on pages 4 to 10, show a recursive process. The sums of the squares may be derived from the sums of the integers, the sums of the cubes from the sums of the squares, and so on. And so, *formula* 2 and *formula* 3, which were derived inductively from these geometric diagrams, are themselves also recursive formulae.

Both these recursive formulae could be used in conjunction with *formula* 1 within Logo programming language to produce formulae for any of the sums of the sums of any power.

Here are general formulae for all the sums of sums of the first four powers (see Appendix 1 for their derivations):

$$^{S}\sum_{i=1}^{n} i^{\,1} = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} \qquad ^{S}\sum_{i=1}^{n} i^{\,2} = \frac{2n+S}{(S+2)!}\times\frac{(n+S)!}{(n-1)!} \qquad ^{S}\sum_{i=1}^{n} i^{\,3} = \frac{6n^2+6Sn+S^2-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!} \qquad ^{S}\sum_{i=1}^{n} i^{\,4} = \frac{24n^3+36Sn^2+14S^2n-10Sn-5S^2+S^3}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}$$

when $S = 1$, these formulae become equivalent to the formulae we are more used to for the sums of these series:

	When $S = 1$:	The formulae we are more used to:
$^{S}\sum_{i=1}^{n} i^{\,1} = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!}$	$^{S=1}\sum_{i=1}^{n} i^{\,1} = \frac{1}{2!}\times n(n+1)$	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
$^{S}\sum_{i=1}^{n} i^{\,2} = \frac{2n+S}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}$	$^{S=1}\sum_{i=1}^{n} i^{\,2} = \frac{2n+1}{3!}\times n(n+1)$	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$^{S}\sum_{i=1}^{n} i^{\,3} = \frac{6n^2+6Sn+S^2-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!}$	$^{S=1}\sum_{i=1}^{n} i^{\,3} = \frac{6n^2+6n}{4!}\times n(n+1)$	$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
$^{S}\sum_{i=1}^{n} i^{\,4} = \frac{24n^3+36Sn^2+14S^2n-10Sn-5S^2+S^3}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}$	$^{S=1}\sum_{i=1}^{n} i^{\,4} = \frac{24n^3+36n^2+4n-4}{5!}\times n(n+1)$	$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Increasingly, my formulae have common factors in the numerator which erode the factorial in the denominator, and so disguise the overall structure of how these formulae have been derived.

The first time this happens is with the sums of the sums of the squares. Whenever S is an even number, $2n$ + S is divisible by 2. The consequences of these events will be passed on to the sums of the sums of the cubes. So, we see that the polynomial in the first sum of the cubes has a common factor of 6, which erodes the factorial to make the sum of the cubes equal to the square of the sum of the integers. Whenever S is divisible by 2, the polynomial for the sums of the sums of the cubes have a common factor of 2. Whenever S is divisible by 3, the polynomial will have a common factor of 3, and so the factorial in the denominator will be eroded.

Similarly, the sums of the fourth and fifth powers show little indication of an overall pattern emerging to explain how such sums are formed, other than the fact that the sum of the integers is always a factor. Here are the conventional formulae for the sums up to the fifth power:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \qquad \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \qquad \sum_{i=1}^{n} i^5 = \frac{n^2(n+1)(2n^3+4n^2+n-1)}{12}$$

No particular pattern emerges either in the formation of the polynomial, or in the number it is divided by. It seems a haphazard business, whereas there is causality and predictability in the 'ancestry' of these sums given by both Alhazen's formula and by mine.

Two general algorithms for the sums of the powers

Two non-recursive general algorithms for the sums of powers may be derived from *formulae 1, 2 & 3* (See the Appendix 2 for these derivations):

from diagram 22

for all $p > 1$

$$ {}^{S}\sum_{i=1}^{n} i^{\,p} = n \times \left({}^{S}\sum_{i=1}^{n} i^{\,p-1} \right) - S \times \left({}^{S+1}\sum_{i=1}^{n-1} i^{\,p-1} \right) \quad - \quad \textit{formula } 2 $$

↓

Algorithm 1
derived from *formulae 1 & 2*:

$$ {}^{S=1}\sum_{i=1}^{n} i^{\,p} = $$

$$ (-1)^{0} \times \frac{n}{1\times 2}^{\,p-1} \times \frac{(n+1)!}{(n-1)!} $$

$$ +(-1)^{1} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-2)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } n \ \& (n-1) \end{array} \right)}{2\times 3} \times \frac{(n+1)!}{(n-2)!} $$

$$ +(-1)^{2} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-3)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } n, (n-1) \ \& \ (n-2) \end{array} \right)}{3\times 4} \times \frac{(n+1)!}{(n-3)!} $$

⇣

$$ +(-1)^{(p-1)} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 2^{nd} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } (n-(p-3)) \end{array} \right)}{(p-2)\times(p-1)} \times \frac{(n+1)!}{(n-p-2)!} $$

$$ +(-1)^{p} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 1^{st} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } (n-(p-2)) \end{array} \right)}{(p-1)\times p} \times \frac{(n+1)!}{(n-p-1)!} $$

$$ +(-1)^{(p+1)} \times \frac{1}{p\times(p+1)} \times \frac{(n+1)!}{(n-p)!} $$

from diagram 21

Alhazen's completed triangle

$$ {}^{S}\sum_{i=1}^{n} i^{\,p+1} = (n+S) \times \left({}^{S}\sum_{i=1}^{n} i^{\,p} \right) - S \times \left({}^{S+1}\sum_{i=1}^{n} i^{\,p} \right) \quad - \quad \textit{formula } 3 $$

↓

Algorithm 2
derived from *formulae 1 & 3*:

$$ {}^{S=1}\sum_{i=1}^{n} i^{\,p} = $$

$$ (-1)^{0} \times \frac{(n+1)^{p-1}}{1\times 2} \times \frac{(n+1)!}{(n-1)!} $$

$$ +(-1)^{1} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-2)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } (n+1) \ \& (n+2) \end{array} \right)}{2\times 3} \times \frac{(n+2)!}{(n-1)!} $$

$$ +(-1)^{2} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-3)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } (n+1), (n+2) \ \& \ (n+3) \end{array} \right)}{3\times 4} \times \frac{(n+3)!}{(n-1)!} $$

⇣

$$ +(-1)^{(p-1)} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 2^{nd} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } (n+p-2) \end{array} \right)}{(p-3)\times(p-2)} \times \frac{(n+p-2)!}{(n-1)!} $$

$$ +(-1)^{p} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 1^{st} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } (n+p-1) \end{array} \right)}{(p-2)\times(p-1)} \times \frac{(n+p-1)!}{(n-1)!} $$

$$ +(-1)^{(p+1)} \times \frac{1}{(p-1)\times p} \times \frac{(n+p)!}{(n-1)!} $$

I hope a reader will able to suggest a more succinct mathematical way to express the verbose algorithms in the brackets, and so turn these two algorithms into two general formulae?

Algorithm 1 describes the sum of a series of factors multiplied by a series of falling factorials.

In this algorithm, these are expressed in this form, $\frac{(n+1)!}{(n-1)!}, \frac{(n+1)!}{(n-2)!}, \frac{(n+1)!}{(n-3)!} \ldots\ldots\ldots \frac{(n+1)!}{(n-p-1)!}, \frac{(n+1)!}{(n-p)!}$; when used in a computer, they should be expressed in this more lengthy form, $n(n+1)$, $(n-1)n(n+1)$, $(n-2)(n-1)n(n+1)$,......etc. to avoid problems arising from divisions by the factorials of negative numbers.

Algorithm 2 is different to *algorithm 1*, in that it shows the sum of a series of factors multiplied by a series of incomplete rising factorials.

Discussion

Algorithm 1 is similar to both Bernoulli and Faulhaber's formulae in that they are both functions of falling factorials, and dissimilar to them in that these two formulae from the history of mathematics are also functions of Bernoulli numbers, whereas I don't think that my algorithm is. I am investigating the question of whether my algorithm is an equivalent algebraic expression to them. Perhaps a reader would like to examine this question also?

Acknowledgements

I had many inspirational teachers when I was an undergraduate at Sheffield Hallam University between 1999 and 2002. In the context of this article, I must single out my history of mathematics tutor David Lingard, who awakened in me a lasting love of his subject – a subject that I am still studying in my retirement eighteen years later. During his course I discovered the two formulae below geometrically in 2000 and included them in my *History of Patchwork Geometry*, which I presented to David for assessment:

$$\sum_{i=1}^{n} i^4 = n\times\sum_{i=1}^{n} i^3 - \sum_{k=1}^{n-1}\sum_{i=1}^{k} i^3 \quad \& \quad \sum_{i=1}^{n} i^5 = n\times\sum_{i=1}^{n} i^4 - \sum_{k=1}^{n-1}\sum_{i=1}^{k} i^4$$

Checking on the internet later I was disappointed to find that I had merely rediscovered Alhazen's formula in a different algebraic form.

David introduced me to Professor Jan van Maanen of Utrecht University when he came to Sheffield in 2002 to deliver a seminar for the History in Mathematics Education Society. Afterwards, we worked for a while by email on a problem he had set at the end of his talk. I got in touch with Jan again last year when I thought I had discovered something interesting, and we worked together by email throughout February and March 2019 on the sum of the sums of the integers and the sum of the squares from a historical perspective. We plan a joint article on the work, but it hasn't been published yet. As you have seen, this work is the starting point for this article.

It has been a privilege to have worked with two such inspirational teachers of the history of mathematics.

My thanks also go to my friend Elizabeth Stocker. Elizabeth is a more highly trained mathematician than I am, and has acted as a proof-reader of various versions of this article over the last year. She has given me much good advice about the formatting and how comprehensible she has found the geometric diagrams, and also pointed out some of my many mistakes. Elizabeth has also become a collaborator during the last two months, and we hope to produce a joint Coda to this article.

JM February 2020

This article was first published on my Jim Milner Sculpture website on 4/8/2020, along with a short summary of it called *Completing Alhazen's Formula*: http://www.jimmilnersculpture.co.uk/maths-completing-alhazens-formula.html

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Victor J. Katz, *A History of Mathematics, An Introduction*. 2nd Ed. 1998. Addison-Wesley

Jan van Maanen and Jim Milner, *Untitled and unpublished paper*, 2019

Jim Milner, *A history of Patchwork Geometry*, Unpublished paper, Sheffield Hallam University, 2000

Contact Jim Milner on:

Website: jimmilnersculpture.co.uk

Email: milner44@btinternet.com

Telephone: +44 (0)1226 763124

Appendix 1

Deriving the formulae below:

$$ {}^{S}\sum_{i=1}^{n} i^{2} = \frac{2n+S}{(S+2)!}\times\frac{(n+S)!}{(n-1)!} \qquad {}^{S}\sum_{i=1}^{n} i^{3} = \frac{6n^{2}+6Sn+S^{2}-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!} \qquad {}^{S}\sum_{i=1}^{n} i^{4} = \frac{\left(24n^{3}+36Sn^{2}+14S^{2}n-10Sn-5S^{2}+S^{3}\right)}{(S+4)!}\times\frac{(n+S)!}{(n-1)!} $$

from: $$ {}^{S}\sum_{i=1}^{n} i^{P} = n\times\left({}^{S}\sum_{i=1}^{n} i^{P-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{P-1}\right) \quad \& \quad {}^{S}\sum_{i=1}^{n} i^{1} = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = n\times\left({}^{S}\sum_{i=1}^{n} i^{1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{1}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = n\times\frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} - S\times\frac{1}{(S+2)!}\times\frac{(n-1+S+1)!}{(n-1-1)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = n\times\frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} - S\times\frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-2)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!}\left(n - S\times\frac{1}{(S+2)}\times\frac{(n-1)}{1}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!}\left(\frac{n(S+2)-S(n-1)}{(S+2)}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{2} = \frac{2n+S}{(S+2)!}\times\frac{(n+S)!}{(n-1)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = n\times\left({}^{S}\sum_{i=1}^{n} i^{2}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{2}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = n\times\frac{(2n+S)}{(S+2)!}\times\frac{(n+S)!}{(n-1)!} - S\times\frac{(2n+S-1)}{(S+3)!}\times\frac{(n+S)!}{(n-2)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}\left(n(2n+S) - \frac{S(2n+S-1)(n-1)}{S+3}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}\left(\frac{\left(2n^{2}+Sn\right)(S+3)-S(2n+S-1)(n-1)}{S+3}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}\left(\frac{2Sn^{2}+6n^{2}+S^{2}n+3Sn-\left(2Sn^{2}+S^{2}n-Sn-2Sn-S^{2}+S\right)}{S+3}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}\left(\frac{6n^{2}+6Sn+S^{2}-S}{S+3}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{6n^{2}+6Sn+S^{2}-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{3} = \frac{6n^{2}+6Sn+S^{2}-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!} $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = n\left({}^{S}\sum_{i=1}^{n} i^{3}\right) - S\left({}^{S+1}\sum_{i=1}^{n-1} i^{3}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = n\left(\frac{6n^{2}+6Sn+S^{2}-S}{(S+3)!}\times\frac{(n+S)!}{(n-1)!}\right) - S\left(\frac{6(n-1)^{2}+6(S+1)(n-1)+(S+1)^{2}-(S+1)}{(S+4)!}\times\frac{(n+S)!}{(n-2)!}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+3)!}\times\frac{(n+S)!}{(n-1)!}\left(\left(\frac{6n^{3}+6Sn^{2}+S^{2}n-Sn}{1}\right) - S(n-1)\left(\frac{6(n-1)^{2}+6(S+1)(n-1)+(S+1)^{2}-(S+1)}{(S+4)}\right)\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}\left(\begin{array}{l}(S+4)\left(6n^{3}+6Sn^{2}+S^{2}n-Sn\right)\\ -S(n-1)\left(6(n-1)^{2}+6(S+1)(n-1)+(S+1)^{2}-(S+1)\right)\end{array}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}\left(\begin{array}{l}\left(6Sn^{3}+6S^{2}n^{2}+S^{3}n-S^{2}n\right)+\left(24n^{3}+24Sn^{2}+4S^{2}n-4Sn\right)\\ -S(n-1)\left(6n^{2}-12n+6+6Sn-6S+6n-6+S^{2}+2S+1-S-1\right)\end{array}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}\left(\begin{array}{l}\left(24n^{3}+6Sn^{3}+24Sn^{2}+6S^{2}n^{2}+S^{3}n+3S^{2}n-4Sn\right)\\ -S(n-1)\left(6n^{2}-6n+6Sn-5S+S^{2}\right)\end{array}\right) $$

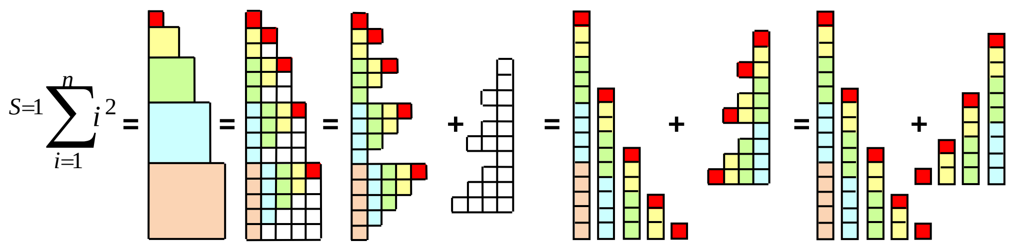
$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}\left(\begin{array}{l}\left(24n^{3}+6Sn^{3}+24Sn^{2}+6S^{2}n^{2}+S^{3}n+3S^{2}n-4Sn\right)\\ -\left(\begin{array}{l}6Sn^{3}-6Sn^{2}+6S^{2}n^{2}-5S^{2}n+S^{3}n\\ -6Sn^{2}+6Sn-6S^{2}n+5S^{2}-S^{3}\end{array}\right)\end{array}\right) $$

$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{1}{(S+4)!}\times\frac{(n+S)!}{(n-1)!}\left(24n^{3}+36Sn^{2}+14S^{2}n-10Sn-5S^{2}+S^{3}\right) $$

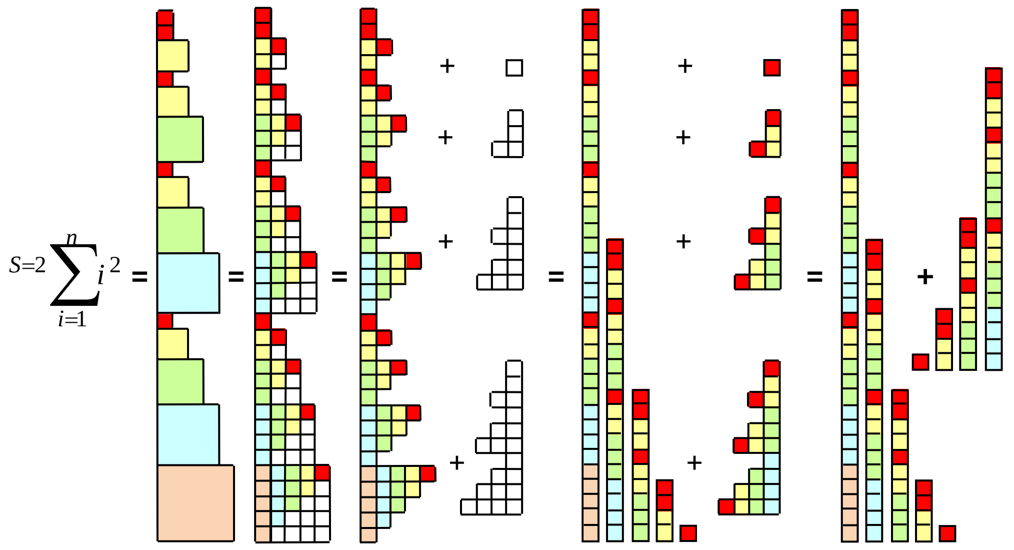
$$ {}^{S}\sum_{i=1}^{n} i^{4} = \frac{\left(24n^{3}+36Sn^{2}+14S^{2}n-10Sn-5S^{2}+S^{3}\right)}{(S+4)!}\times\frac{(n+S)!}{(n-1)!} $$

Pages 4 & 5 show a geometric derivation of the sum of sums of the squares: $\displaystyle {}^{S}\sum_{i=1}^{n} i^2 = n\times\left({}^{S}\sum_{i=1}^{n} i^1\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^1\right)$

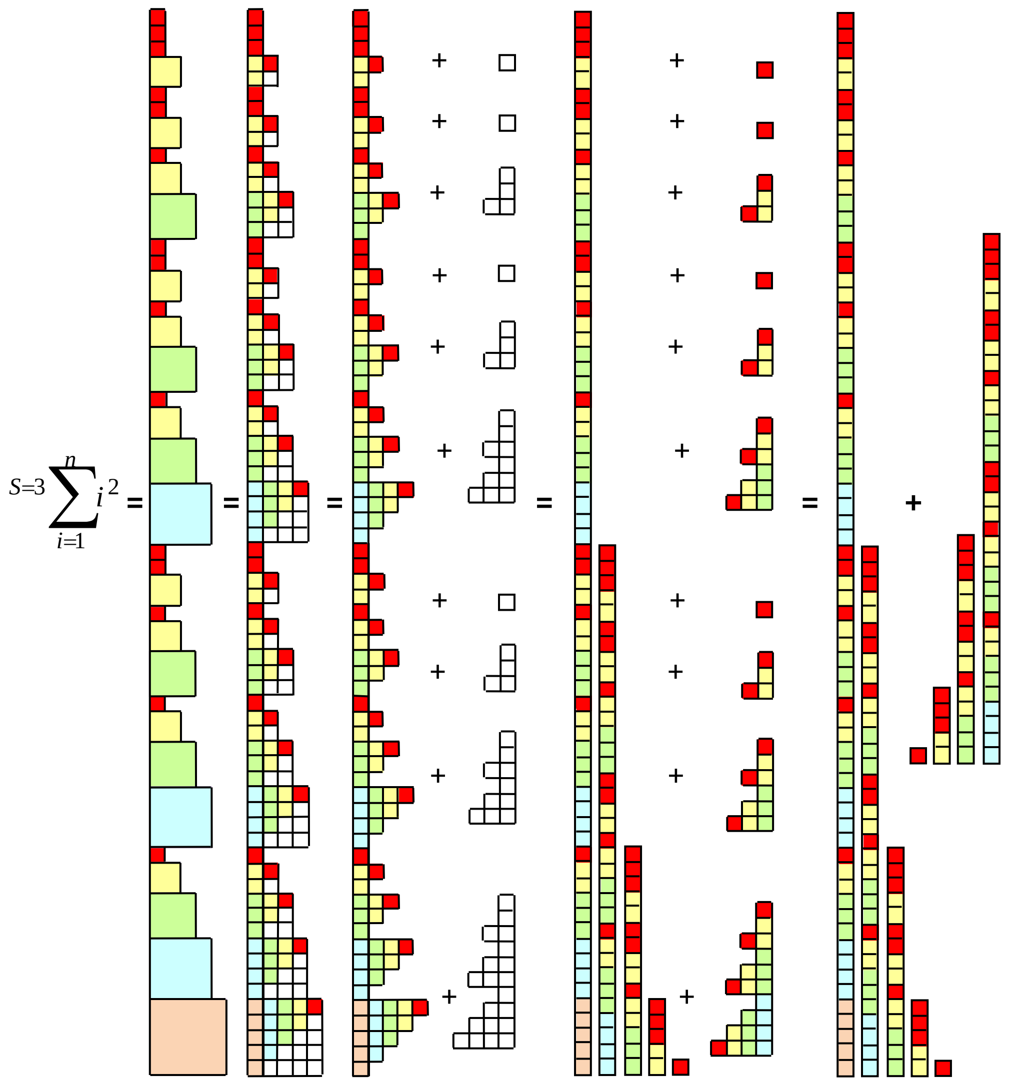
Below there is another, different, geometric derivation which results in a different formula:



$$^{S=1}\sum_{i=1}^{n} i^2 = \left({}^{S=2}\sum_{i=1}^{n} i^1\right) + \left({}^{S=2}\sum_{i=1}^{n-1} i^1\right)$$



$$^{S=2}\sum_{i=1}^{n} i^2 = \left({}^{S=3}\sum_{i=1}^{n} i^1\right) + \left({}^{S=3}\sum_{i=1}^{n-1} i^1\right)$$



$$^{S=3}\sum_{i=1}^{n} i^2 = \left({}^{S=4}\sum_{i=1}^{n} i^1\right) + \left({}^{S=4}\sum_{i=1}^{n-1} i^1\right)$$

by induction

$$^{S}\sum_{i=1}^{n} i^2 = \left({}^{S+1}\sum_{i=1}^{n} i^1\right) + \left({}^{S+1}\sum_{i=1}^{n-1} i^1\right)$$

Substituting: $\displaystyle {}^{S}\sum_{i=1}^{n} i^1 = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!}$

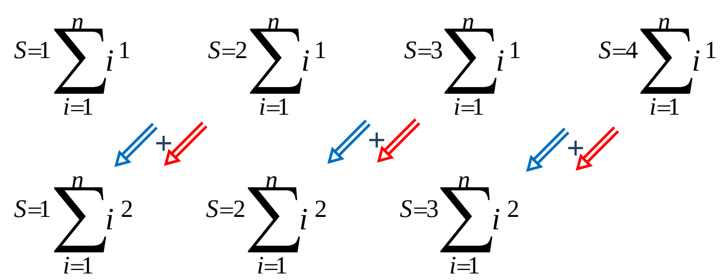
$$^{S}\sum_{i=1}^{n} i^2 = \frac{1}{(S+2)!}\times\frac{(n+S+1)!}{(n-1)!} + \frac{1}{(S+2)!}\times\frac{(n-1+S+1)!}{(n-1-1)!}$$

$$^{S}\sum_{i=1}^{n} i^2 = \frac{1}{(S+2)!}\times\frac{(n+S+1)!}{(n-1)!} + \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-2)!}$$

$$^{S}\sum_{i=1}^{n} i^2 = \frac{1}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}\left((n+S+1)+(n-1)\right)$$

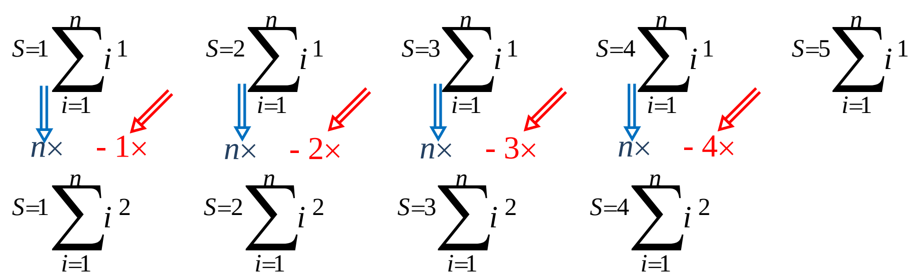
$$^{S}\sum_{i=1}^{n} i^2 = \frac{2n+S}{(S+2)!}\times\frac{(n+S)!}{(n-1)!}$$

$$^{S}\sum_{i=1}^{n} i^2 = \left({}^{S+1}\sum_{i=1}^{n} i^1\right) + \left({}^{S+1}\sum_{i=1}^{n-1} i^1\right)$$



& from pages 4 & 5

$$^{S}\sum_{i=1}^{n} i^2 = n\times\left({}^{S}\sum_{i=1}^{n} i^1\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^1\right)$$



Appendix 2

Ancestry

Diagrammatically, *diagrams 19* and *21* resemble family trees, and so it will be helpful here to think in terms of ancestry in order to produce non-recursive formulae for the sums of powers from them. A study of *diagram 21* shows that the earliest ancestors of the sums of the first five powers are a number of the sums of the sums of the integers.

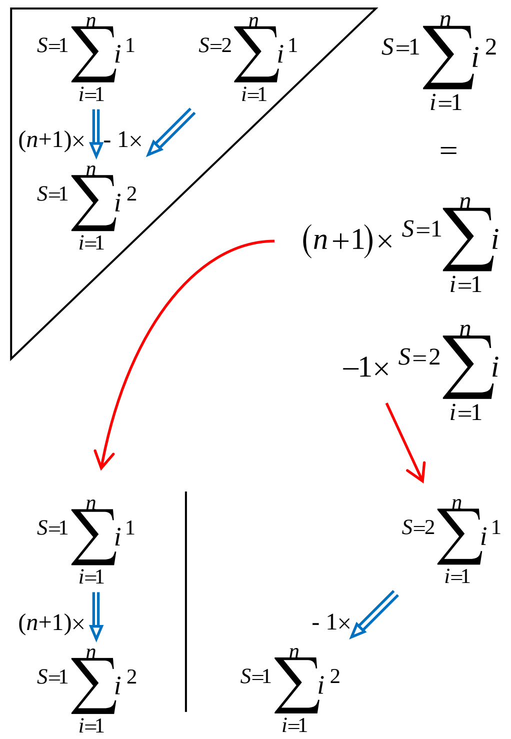
If one were able to draw up a family tree of every single one of our first human ancestors, then we could think of our genetic make-up as being the sum of all our earliest ancestors' genes. Unfortunately, it is never possible to do this completely, and so genealogy is an approximate science. The new science of DNA mapping and analysis is a more exact science these days – we may now bi-pass genealogy to know what our genetic make-up actually is.

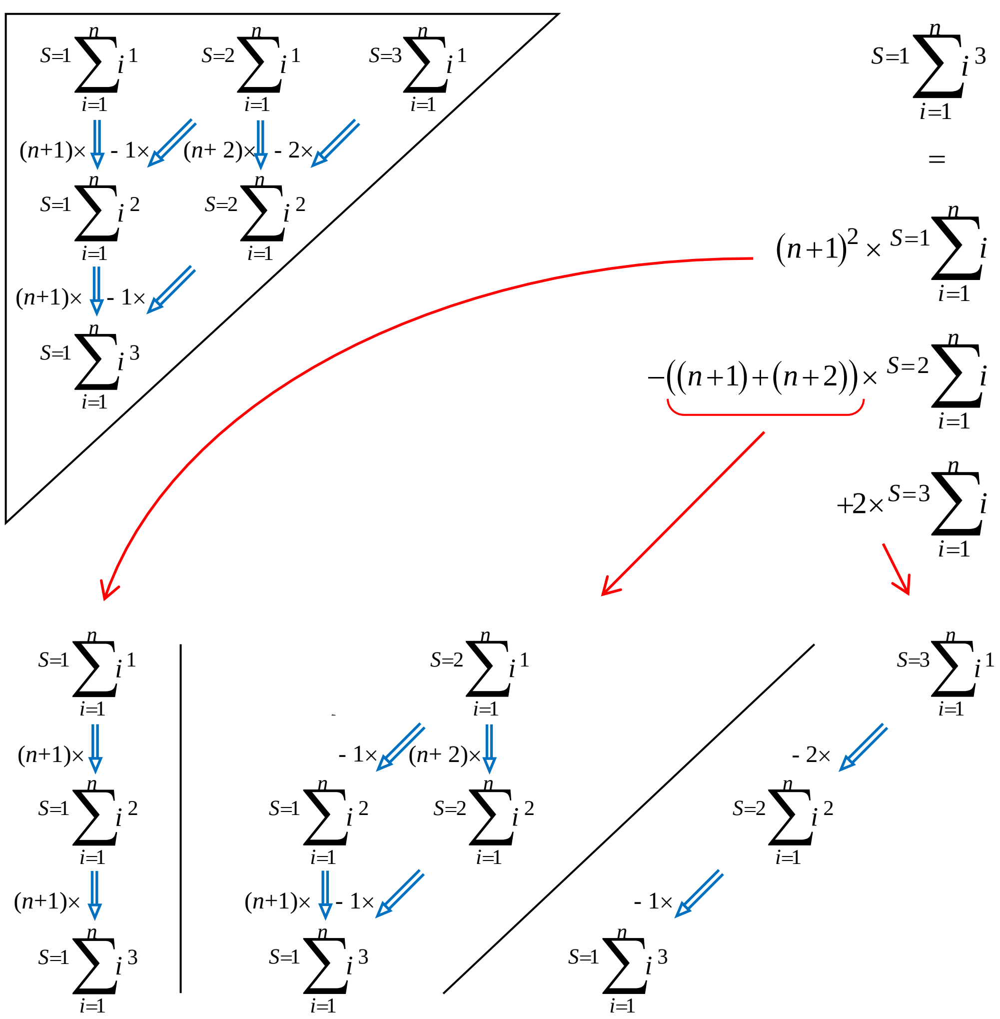
In the following diagrams, the sum of a power is not a sum of genes, but instead the sum of various numerical expressions instead. In this case, we do know who all their original ancestors are. They are all various sums of the sums of integers, who themselves are all descended from the single sum of the integers itself. Rather like the Biblical story that we are all descended from Adam and Eve, and that Eve was in fact created from Adam's rib?

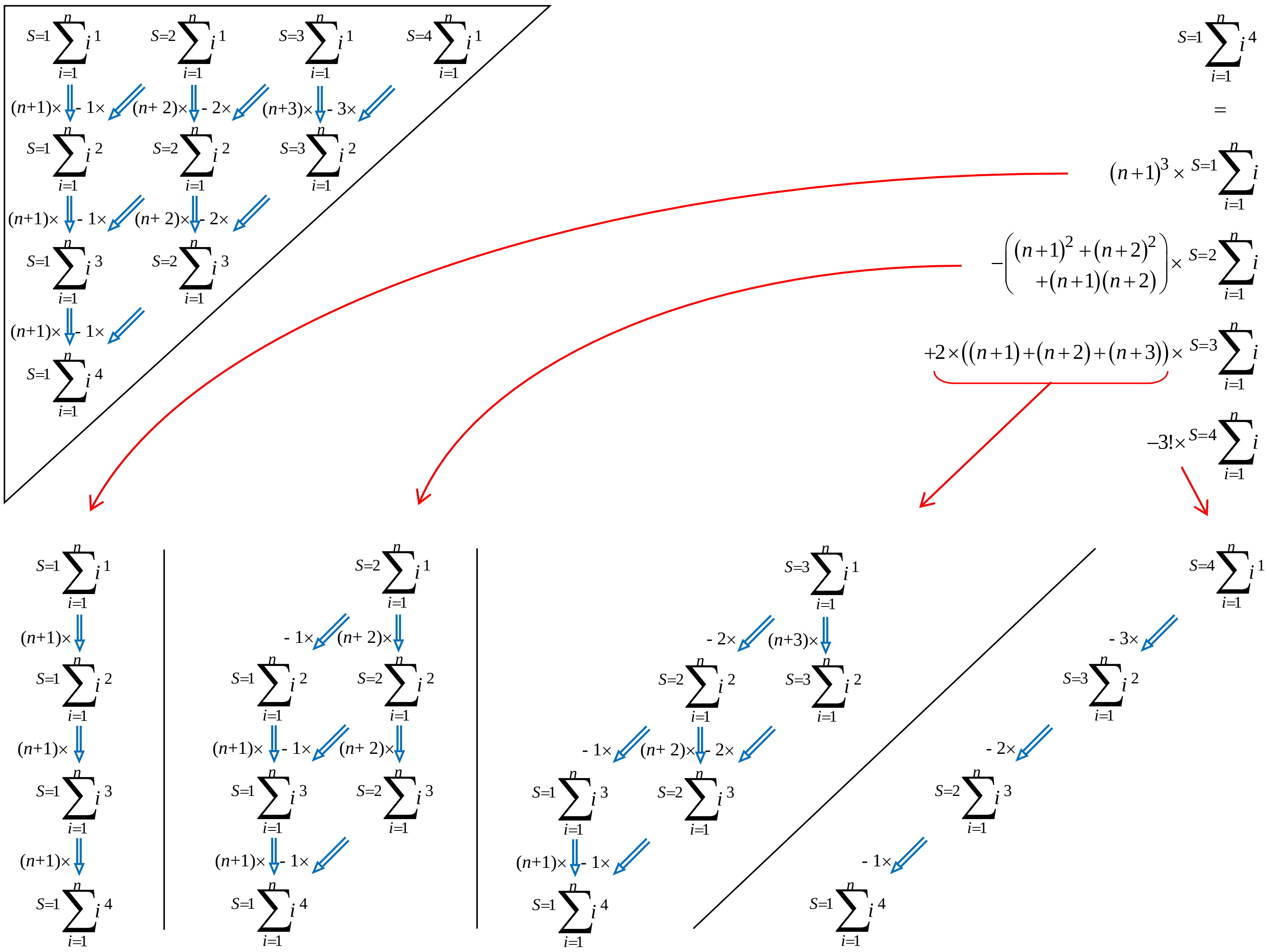
With the sum of a power, incestuous marriages do not occur in their family trees. Their ancestors are not people, but just numbers. The sum of a power is the sum of all the possible routes to their ancestors, the sums of sums of integers. And so finding the sum is simply a matter of looking at their family trees and listing the terms in them. In the diagrams below, I have dissected each family tree into all the descents from each of their sums of the sums of integers ancestors:

from Alhazen's completed triangle *in diagram 21*

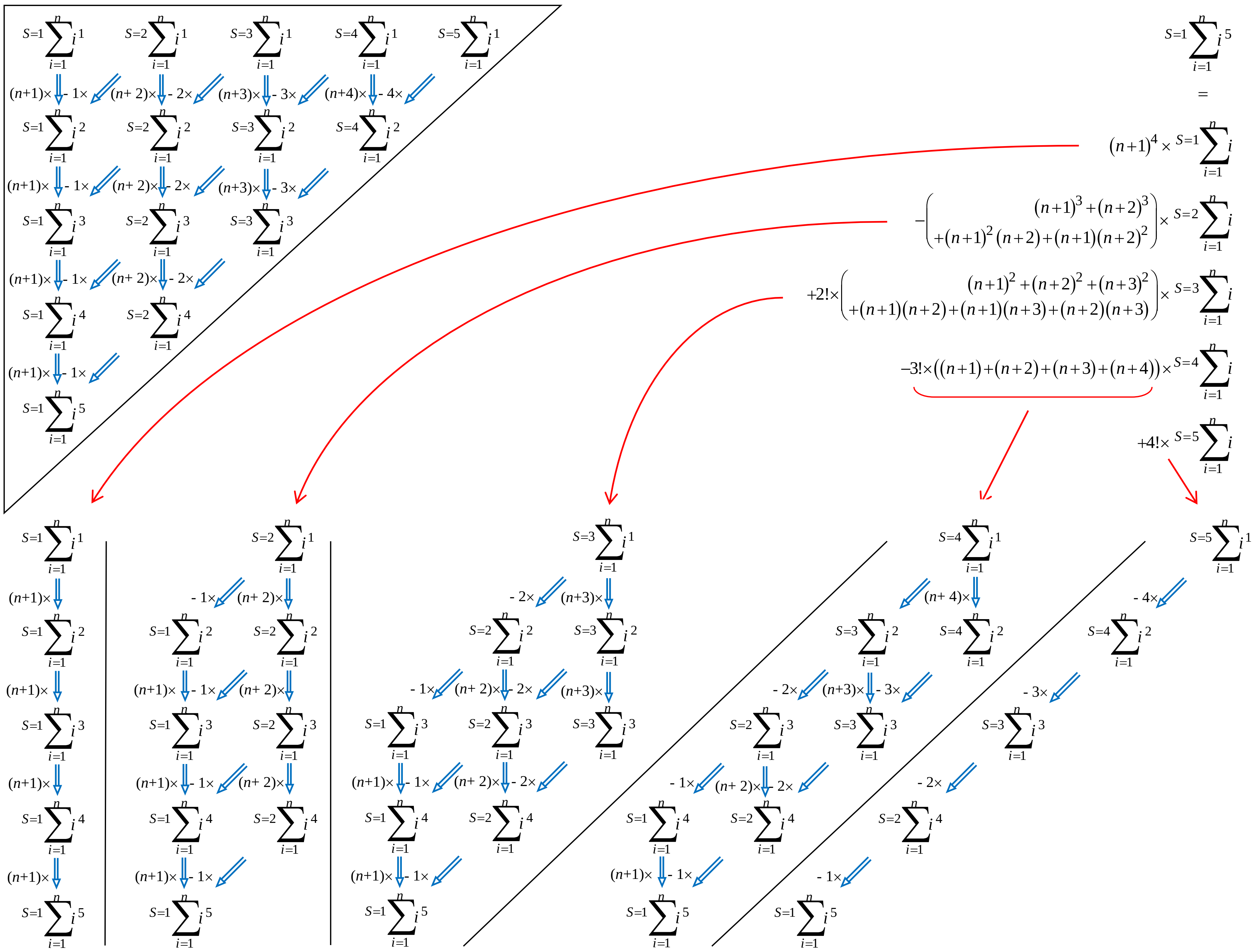
$$ {}^{S}\sum_{i=1}^{n} i^{p+1} = (n+S)\times\left({}^{S}\sum_{i=1}^{n} i^{p}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n} i^{p}\right) $$







$$ {}^{S=1}\sum_{i=1}^{n} i^4 = (n+1)^3 \times {}^{S=1}\sum_{i=1}^{n} i - \left(\begin{matrix} (n+1)^2 + (n+2)^2 \\ +(n+1)(n+2) \end{matrix} \right) \times {}^{S=2}\sum_{i=1}^{n} i + 2\times\big((n+1)+(n+2)+(n+3)\big) \times {}^{S=3}\sum_{i=1}^{n} i - 3! \times {}^{S=4}\sum_{i=1}^{n} i $$



$$ {}^{S=1}\sum_{i=1}^{n} i^5 = (n+1)^4 \times {}^{S=1}\sum_{i=1}^{n} i - \left(\begin{matrix} (n+1)^3 + (n+2)^3 \\ +(n+1)^2(n+2) + (n+1)(n+2)^2 \end{matrix} \right) \times {}^{S=2}\sum_{i=1}^{n} i + 2! \times \left(\begin{matrix} (n+1)^2 + (n+2)^2 + (n+3)^2 \\ +(n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3) \end{matrix} \right) \times {}^{S=3}\sum_{i=1}^{n} i - 3! \times \big((n+1)+(n+2)+(n+3)+(n+4)\big) \times {}^{S=4}\sum_{i=1}^{n} i + 4! \times {}^{S=5}\sum_{i=1}^{n} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^{1} = (n+1)^{0}\times {}^{S=1}\sum_{i=1}^{n} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^{2} = (n+1)^{1}\times {}^{S=1}\sum_{i=1}^{n} i \; -1\times {}^{S=2}\sum_{i=1}^{n} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^{3} = (n+1)^{2}\times {}^{S=1}\sum_{i=1}^{n} i \; -\big((n+1)+(n+2)\big)\times {}^{S=2}\sum_{i=1}^{n} i \; +2!\times {}^{S=3}\sum_{i=1}^{n} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^{4} = (n+1)^{3}\times {}^{S=1}\sum_{i=1}^{n} i \; -\begin{pmatrix}(n+1)^{2}+(n+2)^{2}\\ +(n+1)(n+2)\end{pmatrix}\times {}^{S=2}\sum_{i=1}^{n} i \; +2!\times\big((n+1)+(n+2)+(n+3)\big)\times {}^{S=3}\sum_{i=1}^{n} i \; -3!\times {}^{S=4}\sum_{i=1}^{n} i $$

$$ {}^{S=1}\sum_{i=1}^{n} i^{5} = (n+1)^{4}\times {}^{S=1}\sum_{i=1}^{n} i \; -\begin{pmatrix}(n+1)^{3}+(n+2)^{3}\\ +(n+1)^{2}(n+2)+(n+1)(n+2)^{2}\end{pmatrix}\times {}^{S=2}\sum_{i=1}^{n} i \; +2!\times\begin{pmatrix}(n+1)^{2}+(n+2)^{2}+(n+3)^{2}\\ +(n+1)(n+2)+(n+1)(n+3)+(n+2)(n+3)\end{pmatrix}\times {}^{S=3}\sum_{i=1}^{n} i \; -3!\times\big((n+1)+(n+2)+(n+3)+(n+4)\big)\times {}^{S=4}\sum_{i=1}^{n} i \; +4!\times {}^{S=5}\sum_{i=1}^{n} i $$

$$ \text{where } {}^{S}\sum_{i=1}^{n} i = \frac{1}{(S+1)!}\times\frac{(n+S)!}{(n-1)!} $$

All the sums may be expressed in this way:

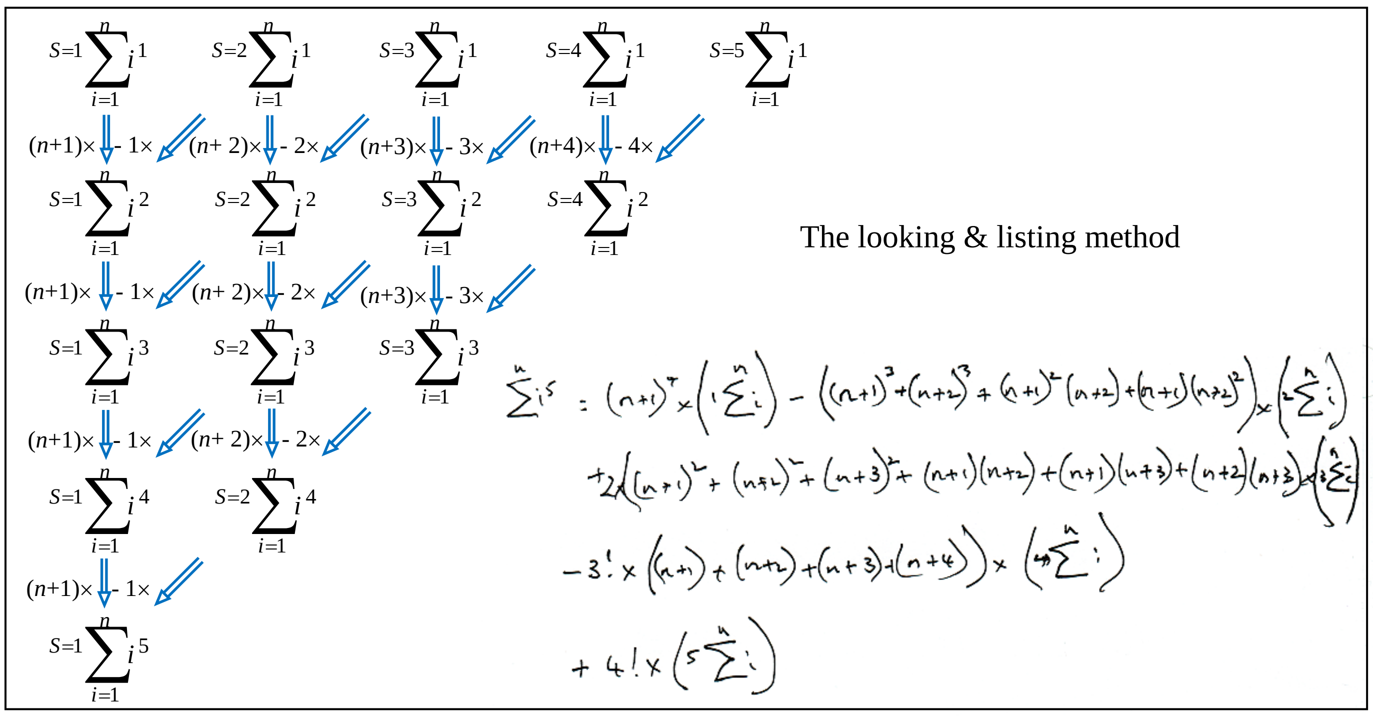
$$ {}^{S=1}\sum_{i=1}^{n} i^{5} = (n+1)^{4}\times {}^{S=1}\sum_{i=1}^{n} i \; -\begin{pmatrix}\textit{The sum of all the } 3^{rd} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors } (n+1)\,\&\,(n+2)\end{pmatrix}\times {}^{S=2}\sum_{i=1}^{n} i \; +2!\times\begin{pmatrix}\textit{The sum of all the } 2^{nd} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors } (n+1),(n+2)\,\&\,(n+3)\end{pmatrix}\times {}^{S=3}\sum_{i=1}^{n} i \; -3!\times\begin{pmatrix}\textit{The sum of all the } 1^{st} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors}(n+1),(n+2),(n+3)\&(n+4)\end{pmatrix}\times {}^{S=4}\sum_{i=1}^{n} i \; +4!\times {}^{S=5}\sum_{i=1}^{n} i $$

So, by induction:

$$ {}^{S=1}\sum_{i=1}^{n} i^{p} = (n+1)^{p-1}\times {}^{S=1}\sum_{i=1}^{n} i \; +(-1)^{1}\times 1!\times\begin{pmatrix}\textit{The sum of all the } (p-2)^{th} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors } (n+1)\,\&\,(n+2)\end{pmatrix}\times {}^{S=2}\sum_{i=1}^{n} i \; +(-1)^{2}\times 2!\times\begin{pmatrix}\textit{The sum of all the } (p-3)^{th} \textit{ order polynomials}\\ \textit{produced by every possible multiplicaton}\\ \textit{of the factors } (n+1),(n+2)\,\&\,(n+3)\end{pmatrix}\times {}^{S=3}\sum_{i=1}^{n} i $$

$$ \vdots $$

$$ +(-1)^{(p-1)}\times(p-3)!\times\begin{pmatrix}\textit{The sum of all the } 2^{nd} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors from } (n+1) \textit{ to } (n+p-2)\end{pmatrix}\times {}^{S=(p-2)}\sum_{i=1}^{n} i \; +(-1)^{(p)}\times(p-2)!\times\begin{pmatrix}\textit{The sum of all the } 1^{st} \textit{ order polynomials}\\ \textit{produced by every possible multiplication}\\ \textit{of the factors from } (n+1) \textit{ to } (n+p-1)\end{pmatrix}\times {}^{S=(p-1)}\sum_{i=1}^{n} i \; +(-1)^{(p+1)}\times(p-1)!\times {}^{S=p}\sum_{i=1}^{n} i $$



The method of deriving a formula by simply looking at a diagram and then listing the descents is certainly less long-winded than the method of dissecting a diagram into all the descents from each individual sum of the sums of the integers ancestor. However, the dissected diagrams show graphically why successive terms have an increasing number of factors, and also why they are the sums of all the possible polynomials of decreasing orders created by every possible multiplications of these factors. In effect, it justifies the wording of the algorithm above, and so I have thought them worth including here.

substituting: $$^{S}\sum_{i=1}^{n} i = \frac{1}{(S+1)!} \times \frac{(n+S)!}{(n-1)!}$$

$$^{S=1}\sum_{i=1}^{n} i^{p}$$

$$=$$

$$(n+1)^{p-1} \times {}^{S=1}\sum_{i=1}^{n} i$$

$$+(-1)^{1} \times 1! \times \left(\begin{array}{c} \textit{The sum of all the } (p-2)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } (n+1) \,\&\, (n+2) \end{array} \right) \times {}^{S=2}\sum_{i=1}^{n} i$$

$$+(-1)^{2} \times 2! \times \left(\begin{array}{c} \textit{The sum of all the } (p-3)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplicaton} \\ \textit{of the factors } (n+1),(n+2) \,\&\, (n+3) \end{array} \right) \times {}^{S=3}\sum_{i=1}^{n} i$$

$$\vdots$$

$$+(-1)^{(p-1)} \times (p-3)! \times \left(\begin{array}{c} \textit{The sum of all the } 2^{nd} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } (n+1) \textit{ to } (n+p-2) \end{array} \right) \times {}^{S=(p-2)}\sum_{i=1}^{n} i$$

$$+(-1)^{(p)} \times (p-2)! \times \left(\begin{array}{c} \textit{The sum of all the } 1^{st} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } (n+1) \textit{ to } (n+p-1) \end{array} \right) \times {}^{S=(p-1)}\sum_{i=1}^{n} i$$

$$+(-1)^{(p+1)} \times (p-1)! \times {}^{S=p}\sum_{i=1}^{n} i$$

$$^{S=1}\sum_{i=1}^{n} i^{p}$$

$$=$$

$$(-1)^{0} \times \frac{(n+1)^{p-1}}{1 \times 2} \times \frac{(n+1)!}{(n-1)!}$$

$$+(-1)^{1} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-2)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } (n+1) \,\&\, (n+2) \end{array} \right)}{2 \times 3} \times \frac{(n+2)!}{(n-1)!}$$

$$+(-1)^{2} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } (p-3)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } (n+1),(n+2) \,\&\, (n+3) \end{array} \right)}{3 \times 4} \times \frac{(n+3)!}{(n-1)!}$$

$$\vdots$$

$$+(-1)^{(p-1)} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 2^{nd} \textit{ order polynomials} \\ \textit{produced every possible multiplication} \\ \textit{of the factors from n to } (n+p-2) \end{array} \right)}{(p-3) \times (p-2)} \times \frac{(n+p-2)!}{(n-1)!}$$

$$+(-1)^{p} \times \frac{\left(\begin{array}{c} \textit{The sum of all the } 1^{st} \textit{ order polynomials} \\ \textit{produced every possible multiplication} \\ \textit{of the factors from n to } (n+p-1) \end{array} \right)}{(p-2) \times (p-1)} \times \frac{(n+p-1)!}{(n-1)!}$$

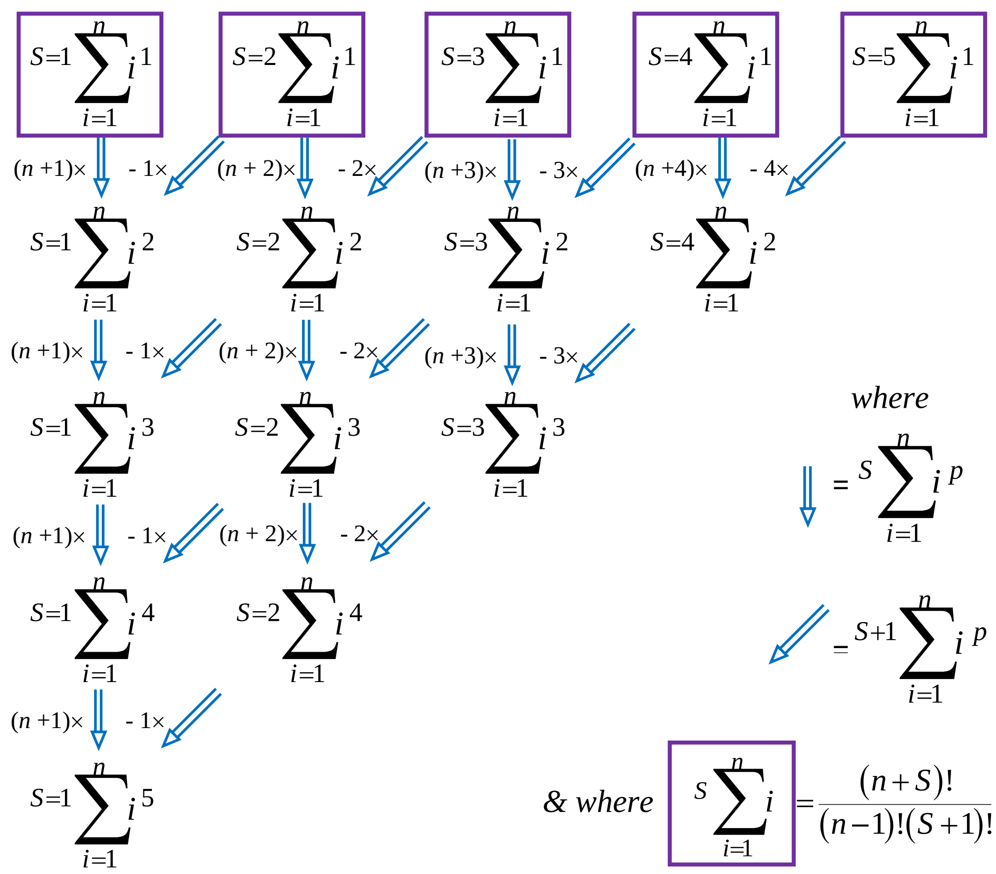
$$+(-1)^{(p+1)} \times \frac{1}{(p-1) \times p} \times \frac{(n+p)!}{(n-1)!}$$

Diagram 21

Alhazen's completed triangle

$$^{S}\sum_{i=1}^{n} i^{p+1} = (n+S) \times \left({}^{S}\sum_{i=1}^{n} i^{p} \right) - S \times \left({}^{S+1}\sum_{i=1}^{n} i^{p} \right)$$ - *formula 3*

Algorithm 2, derived from *formula 3*:



where

$$\Downarrow \; = {}^{S}\sum_{i=1}^{n} i^{p}$$

$$\swarrow \; = {}^{S+1}\sum_{i=1}^{n} i^{p}$$

& where $$^{S}\sum_{i=1}^{n} i = \frac{(n+S)!}{(n-1)!(S+1)!}$$

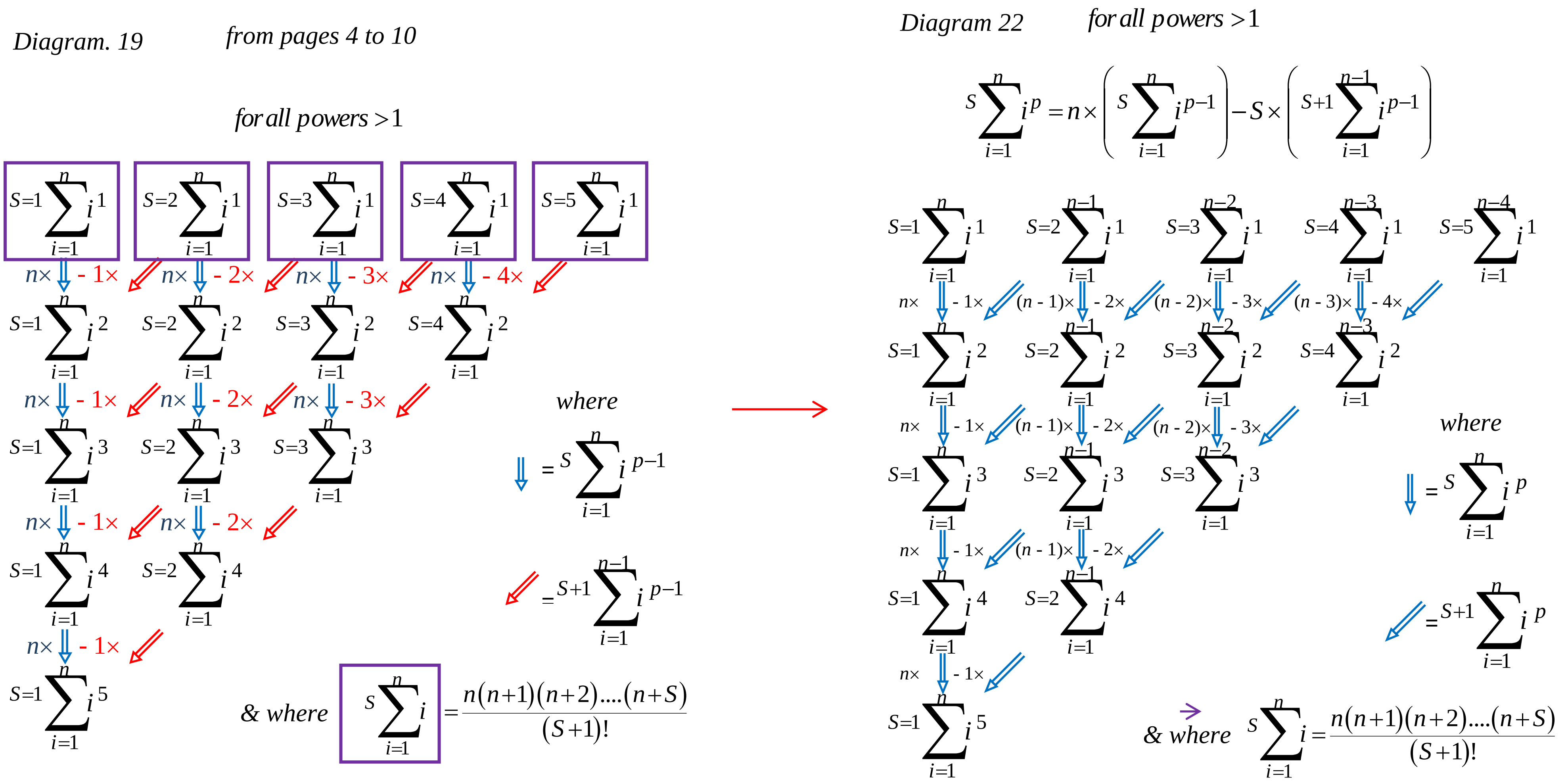
Ancestry 2

Finding a non-recursive algorithm for the sums of powers by studying the ancestry of *diagram 19* is more complicated than from *diagram 21*. *Diagram 19* was built from the top down, to show how the sums of sums of powers could be derived from the sums of sums of integers. From the perspective of the fifth power, the first descent from the sum of the integers remains as it is shown, but the descent of the sum of the fifth power from the sum of the sums of the fourth

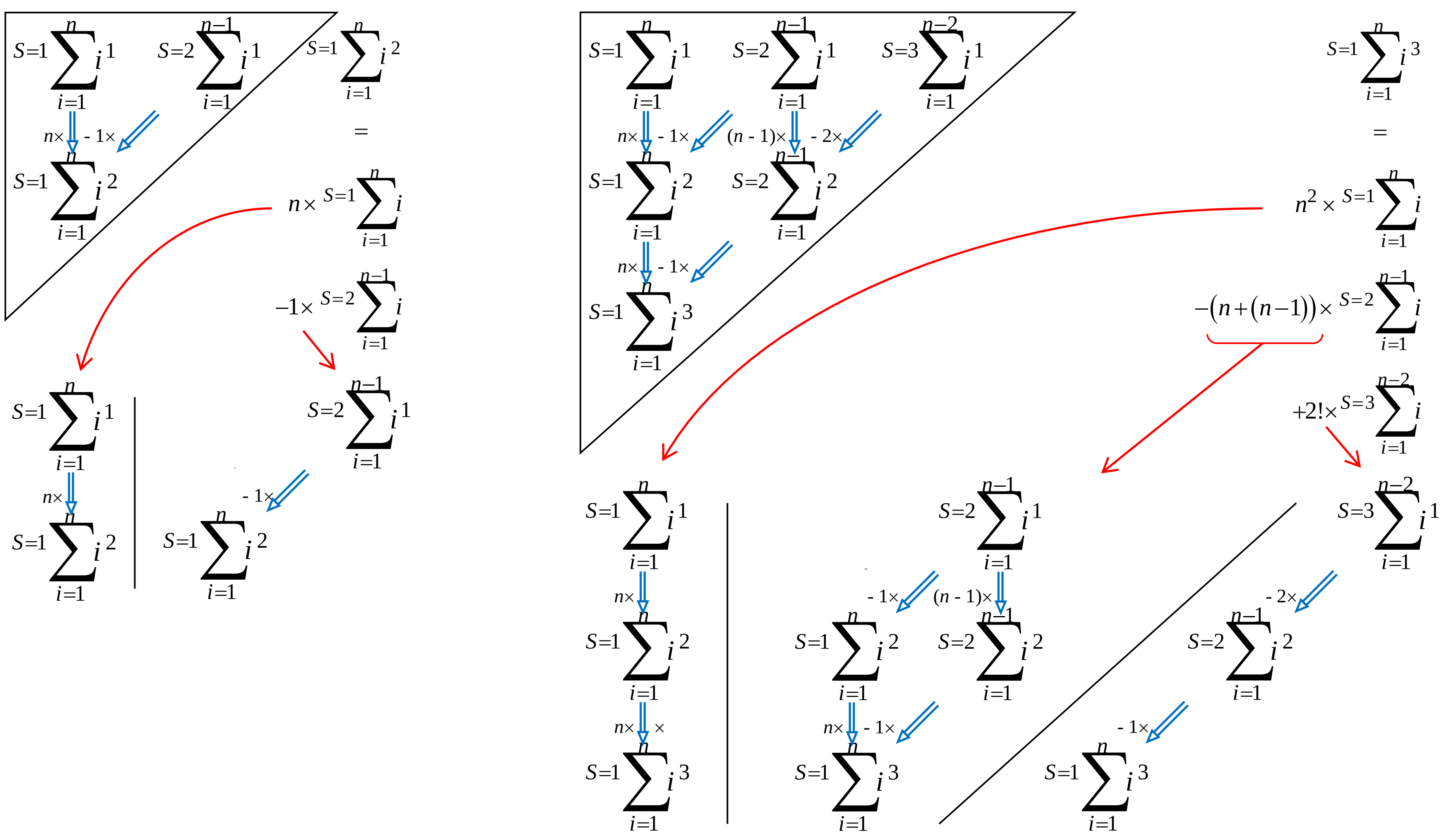
power is not from ${}^{S=2}\sum_{i=1}^{n} i^{4}$, but from ${}^{S=2}\sum_{i=1}^{n-1} i^{4}$, and ${}^{S=2}\sum_{i=1}^{n-1} i^{4}$ is not descended from ${}^{S=2}\sum_{i=1}^{n} i^{1}$ in the vertical descent, but from ${}^{S=2}\sum_{i=1}^{n-1} i^{1}$ with multiplications of $(n-1)$ every generation, not n.

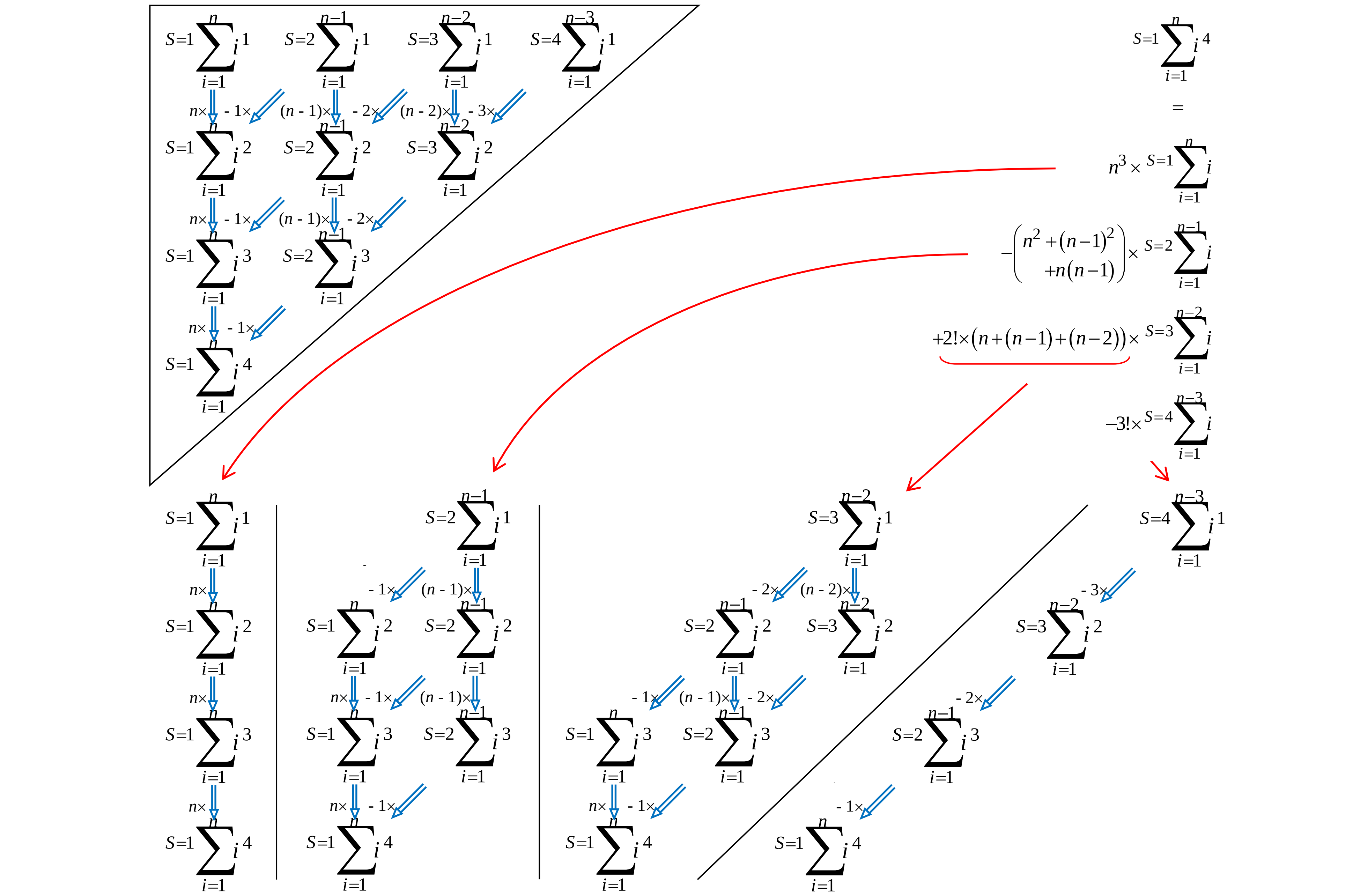
Similarly, the other descent to ${}^{S=2}\sum_{i=1}^{n-1} i^{4}$ is not from ${}^{S=3}\sum_{i=1}^{n} i^{3}$, but from ${}^{S=3}\sum_{i=1}^{n-2} i^{3}$ which is itself descended from ${}^{S=3}\sum_{i=1}^{n-2} i^{1}$ with multiplications of $(n-2)$ every generation, not n.

And so on. So we may re-draw *diagram 19*, from the powers' perspective of their lineage as in *diagram 22*:

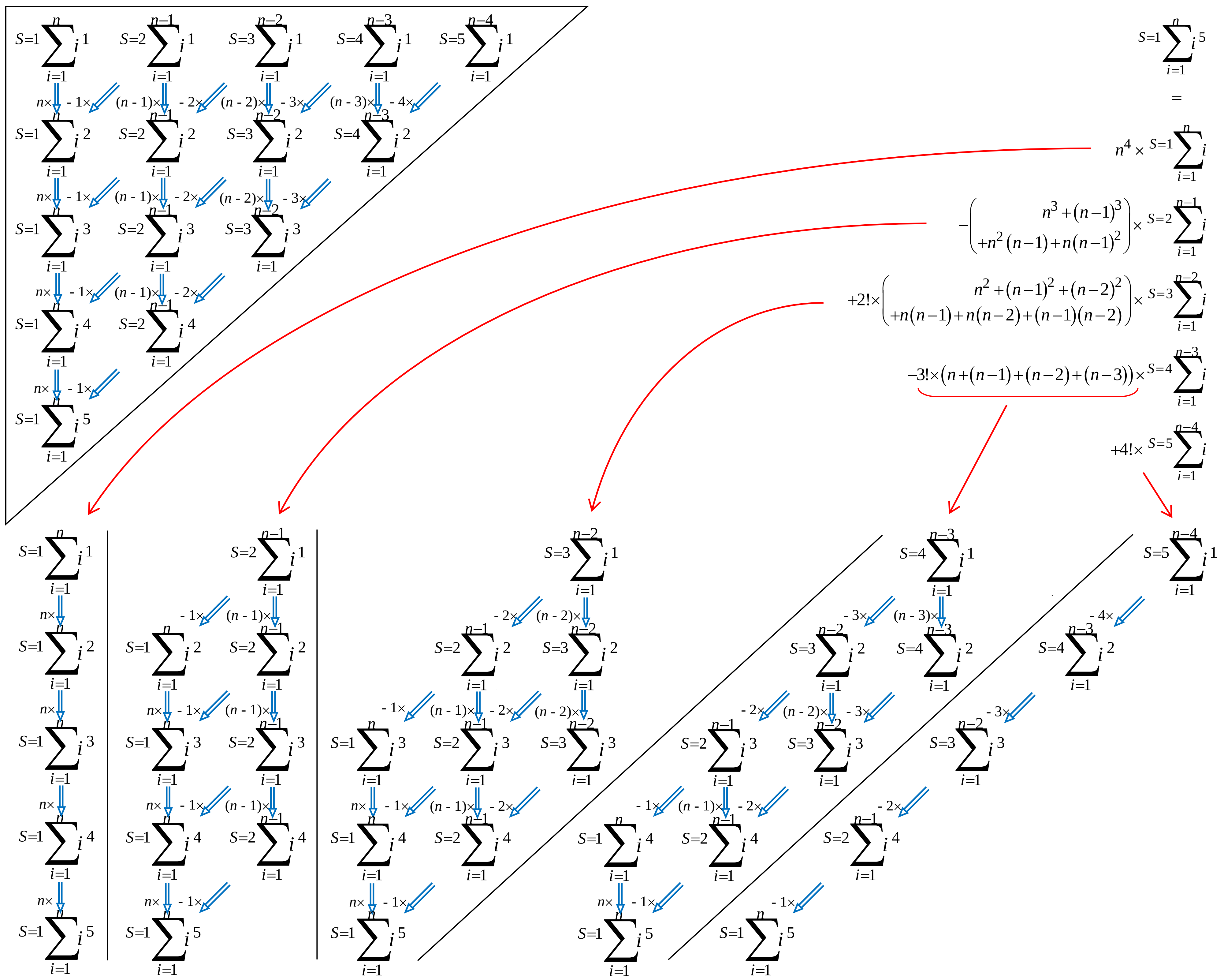


and so may now derive formulae for the sums of the first five powers by dissecting *diagram 22:*





$$ {}^{S=1}\sum_{i=1}^{n} i^4 = n^3 \times {}^{S=1}\sum_{i=1}^{n} i - \begin{pmatrix} n^2+(n-1)^2 \\ +n(n-1) \end{pmatrix} \times {}^{S=2}\sum_{i=1}^{n-1} i + 2!\times\big(n+(n-1)+(n-2)\big)\times {}^{S=3}\sum_{i=1}^{n-2} i - 3!\times {}^{S=4}\sum_{i=1}^{n-3} i $$



$$ {}^{S=1}\sum_{i=1}^{n} i^5 = n^4 \times {}^{S=1}\sum_{i=1}^{n} i - \begin{pmatrix} n^3+(n-1)^3 \\ +n^2(n-1)+n(n-1)^2 \end{pmatrix} \times {}^{S=2}\sum_{i=1}^{n-1} i + 2!\times\begin{pmatrix} n^2+(n-1)^2+(n-2)^2 \\ +n(n-1)+n(n-2)+(n-1)(n-2) \end{pmatrix} \times {}^{S=3}\sum_{i=1}^{n-2} i - 3!\times\big(n+(n-1)+(n-2)+(n-3)\big)\times {}^{S=4}\sum_{i=1}^{n-3} i + 4!\times {}^{S=5}\sum_{i=1}^{n-4} i $$

$${}^{S=1}\sum_{i=1}^{n} i = n^0 \times {}^{S=1}\sum_{i=1}^{n} i$$

$${}^{S=1}\sum_{i=1}^{n} i^2 = n \times {}^{S=1}\sum_{i=1}^{n} i - 1 \times {}^{S=2}\sum_{i=1}^{n-1} i$$

$${}^{S=1}\sum_{i=1}^{n} i^3 = n^2 \times {}^{S=1}\sum_{i=1}^{n} i - \left(n+(n-1)\right) \times {}^{S=2}\sum_{i=1}^{n-1} i + 2! \times {}^{S=3}\sum_{i=1}^{n-2} i$$

$${}^{S=1}\sum_{i=1}^{n} i^4 = n^3 \times {}^{S=1}\sum_{i=1}^{n} i - \begin{pmatrix} n^2+(n-1)^2 \\ +n(n-1) \end{pmatrix} \times {}^{S=2}\sum_{i=1}^{n-1} i + 2! \times \left(n+(n-1)+(n-2)\right) \times {}^{S=3}\sum_{i=1}^{n-2} i - 3! \times {}^{S=4}\sum_{i=1}^{n-3} i$$

$${}^{S=1}\sum_{i=1}^{n} i^5 = n^4 \times {}^{S=1}\sum_{i=1}^{n} i - \begin{pmatrix} n^3+(n-1)^3 \\ +n^2(n-1)+n(n-1)^2 \end{pmatrix} \times {}^{S=2}\sum_{i=1}^{n-1} i + 2! \times \begin{pmatrix} n^2+(n-1)^2+(n-2)^2 \\ +n(n-1)+n(n-2)+(n-1)(n-2) \end{pmatrix} \times {}^{S=3}\sum_{i=1}^{n-2} i - 3! \times \left(n+(n-1)+(n-2)+(n-3)\right) \times {}^{S=4}\sum_{i=1}^{n-3} i + 4! \times {}^{S=5}\sum_{i=1}^{n-4} i$$

---

substituting: $${}^{S}\sum_{i=1}^{n} i = \frac{1}{(S+1)!} \times \frac{(n+S)!}{(n-1)!}$$

$${}^{S=1}\sum_{i=1}^{n} i^1 = \frac{n^0}{1\times 2} \times \frac{(n+1)!}{(n-1)!}$$

$${}^{S=1}\sum_{i=1}^{n} i^2 = \frac{n}{1\times 2} \times \frac{(n+1)!}{(n-1)!} - \frac{1}{2\times 3} \times \frac{(n+1)!}{(n-2)!}$$

$${}^{S=1}\sum_{i=1}^{n} i^3 = \frac{n^2}{1\times 2} \times \frac{(n+1)!}{(n-1)!} - \frac{\left(n+(n-1)\right)}{2\times 3} \times \frac{(n+1)!}{(n-2)!} + \frac{1}{3\times 4} \times \frac{(n+1)!}{(n-3)!}$$

$${}^{S=1}\sum_{i=1}^{n} i^4 = n^3 \times \frac{1}{2!} \times \frac{(n+1)!}{(n-1)!} - \begin{pmatrix} n^2+(n-1)^2 \\ +n(n-1) \end{pmatrix} \frac{1}{3!} \times \frac{(n+1)!}{(n-2)!} + 2! \times \left(n+(n-1)+(n-2)\right) \times \frac{1}{4!} \times \frac{(n+1)!}{(n-3)!} - 3! \times \frac{1}{5!} \times \frac{(n+1)!}{(n-4)!}$$

$${}^{S=1}\sum_{i=1}^{n} i^5 = n^4 \times \frac{1}{2!} \times \frac{(n+1)!}{(n-1)!} - \begin{pmatrix} n^3+(n-1)^3 \\ +n^2(n-1)+n(n-1)^2 \end{pmatrix} \frac{1}{3!} \times \frac{(n+1)!}{(n-2)!} + 2! \times \begin{pmatrix} n^2+(n-1)^2+(n-2)^2 \\ +n(n-1)+n(n-2)+(n-1)(n-2) \end{pmatrix} \frac{1}{4!} \times \frac{(n+1)!}{(n-3)!} - 3! \times \left(n+(n-1)+(n-2)+(n-3)\right) \times \frac{1}{5!} \times \frac{(n+1)!}{(n-4)!} + 4! \times \frac{1}{6!} \times \frac{(n+1)!}{(n-5)!}$$

by induction:

$$\left.\begin{aligned}
&{}^{S=1}\sum_{i=1}^{n} i^p \\
&= \\
&(-1)^0 \times \frac{n}{1\times 2}^{\,p-1} \times \frac{(n+1)!}{(n-1)!} \\
&+(-1)^1 \times \frac{\begin{pmatrix} \textit{The sum of all the } (p-2)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } n \,\&\, (n-1) \end{pmatrix}}{2\times 3} \times \frac{(n+1)!}{(n-2)!} \\
&+(-1)^2 \times \frac{\begin{pmatrix} \textit{The sum of all the } (p-3)^{th} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors } n, (n-1) \,\&\, (n-2) \end{pmatrix}}{3\times 4} \times \frac{(n+1)!}{(n-3)!} \\
&\quad\vdots \\
&+(-1)^{(p-1)} \times \frac{\begin{pmatrix} \textit{The sum of all the } 2^{nd} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } \left(n-(p-3)\right) \end{pmatrix}}{(p-2)\times(p-1)} \times \frac{(n+1)!}{(n-p-2)!} \\
&+(-1)^{p} \times \frac{\begin{pmatrix} \textit{The sum of all the } 1^{st} \textit{ order polynomials} \\ \textit{produced by every possible multiplication} \\ \textit{of the factors from } n \textit{ to } \left(n-(p-2)\right) \end{pmatrix}}{(p-1)\times p} \times \frac{(n+1)!}{(n-p-1)!} \\
&+(-1)^{(p+1)} \times \frac{1}{p\times(p+1)} \times \frac{(n+1)!}{(n-p)!}
\end{aligned}\right\}$$

Algorithm 1, derived from *formula* 2

Postscript

The sums of sums of i^p, where p = 0

Abstract

$${}^{S}\sum_{i=1}^{n} i = \frac{n(n+1)(n+2)....(n+S)}{(S+1)!} = \frac{(n+S)!}{(n-1)!(S+1)!}$$

Pages 2 and 3 of *Sums of Sums of Power Series* demonstrate geometrical that the formula above for the sums of sums of the integers is correct for $S = 1$ to $S = 5$, and so by induction is true for all S.

Pages 4 to 10 demonstrate geometric proofs of the formula below for fifteen sums of sums of power series between $p = 2$ and $p = 5$:

$$for\,all\ p>1$$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right)$$

I will show below that this formula is also true for $p = 1$, which is to say that there is another way to derive the sums of sums of integers, this time from the sums of the sums of the power series for $p = 0$.

When $p = 0$, i^p always equals 1, so $\sum_{i=1}^{n} i^0 = \left({}^{S=1}\sum_{i=1}^{n} i^0\right) = n$. The sum of the sums of p = 0, $\sum_{j}^{n}\sum_{i=1}^{j} i^0 = \left({}^{S=2}\sum_{i=1}^{n} i^0\right) = \left({}^{S=1}\sum_{n=1}^{n} n\right) = \left({}^{S=1}\sum_{i=1}^{n} i\right)$.

In general, $\left({}^{S}\sum_{i=1}^{n} i^0\right) = \left({}^{S-1}\sum_{i=1}^{n} i^1\right)$ and so $${}^{S}\sum_{i=1}^{n} i^0 = \frac{n(n+1)(n+2)....(n+S-1)}{S!} = \frac{(n+S-1)!}{(n-1)!S!}$$

Here is an Excel spreadsheet demonstration of this:

		S					
n	***n* ^0**	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210

These are the sums of sums of *i* ^0

Here are the Sums of the Sums of the integers derived from the Sums of the Sums of the *i* = 0 using the formula below:

$${}^{S}\sum_{i=1}^{n} i^{1} = n\times\left({}^{S}\sum_{i=1}^{n} i^{0}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{0}\right)$$

1		1	1	1	1	1
2		3	4	5	6	7
3		6	10	15	21	28
4		10	20	35	56	84
5		15	35	70	126	210

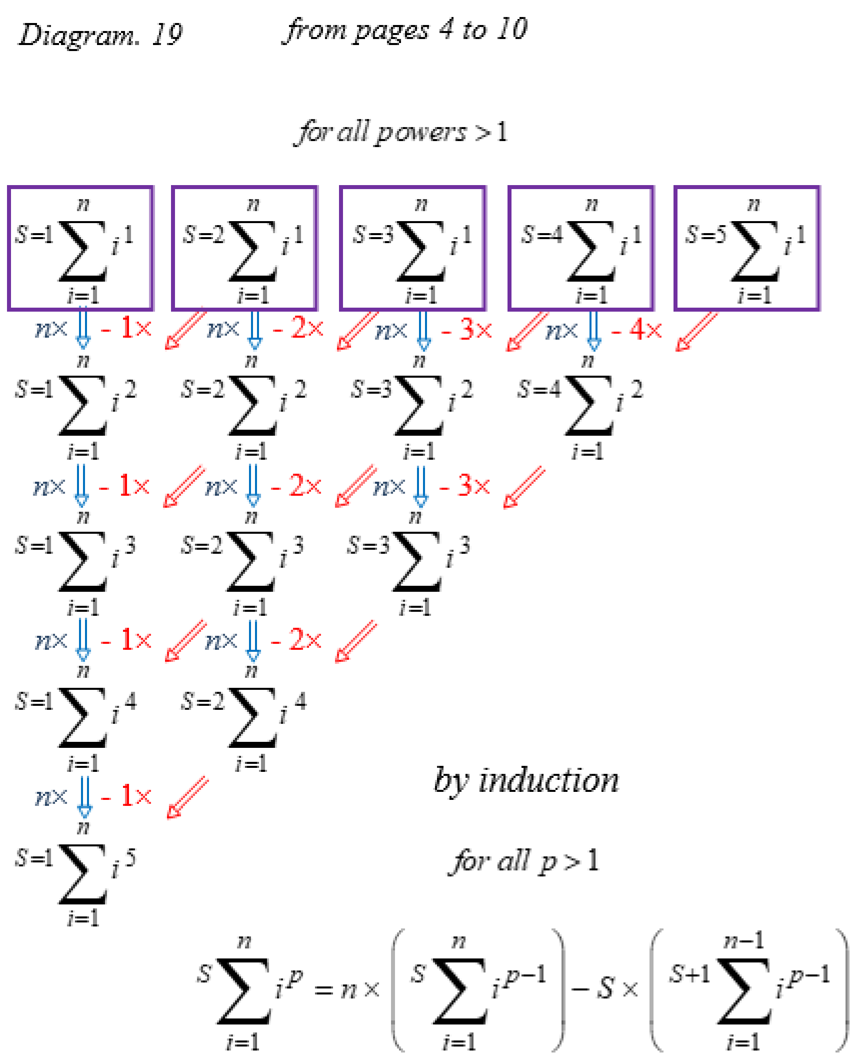
This is correct, so now we know the formula is also true for $p = 1$.

$$for\,all\ p>0$$

$${}^{S}\sum_{i=1}^{n} i^{p} = n\times\left({}^{S}\sum_{i=1}^{n} i^{p-1}\right) - S\times\left({}^{S+1}\sum_{i=1}^{n-1} i^{p-1}\right)$$

Jim Milner · www.jimmilnersculpture.co.uk · +44 (0)1226 763124

Here is *Diagram 19* from page 12 of *Sums of Sums of Power Series*:



We may now expand it to include the sums of sums of i^{0} :